THE STORY OF THE BINOMIAL THEOREM

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1. The early period. The Binomial Theorem, familiar at least in its elementary aspects to every student of algebra, has a long and reasonably plain history. Most people associate it vaguely in their minds with the name of Newton; he either invented it or it was carved on his tomb. In some way or other it was his theorem. Well, as a matter of fact it wasn't, although his work did mark an important advance in the general theory.

We find the first trace of the Binomial Theorem in Euclid II, 4, "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle of the segments." If the segments are a and b this means in algebraic language

$$(a+b)^2 = a^2 + b^2 + 2ab. \tag{1}$$

The corresponding formula for the square of a difference is found in Euclid II, 7, "If a straight line be cut at random, the square on the whole and that on one of the segments both together, are equal to twice the rectangle contained by the whole and said segment, and the square on the remaining segment."

Here if a represents the whole, and b the first segment, we have

$$a^2 + b^2 = 2ab + (a-b)^2 \tag{2}$$

It would have been perfectly easy for Euclid to go ahead and prove the formula for the cube of a binomial, but that would have broken the thread of the argument. In Books II and X he was prodigiously interested in the squares of binomials, any generalization of these does not seem to have interested him at all. The modern tendency to generalize as far as possible, and stretch each theorem to its most general form, was quite foreign to the thinking of the Greeks in mathematics; clearness and precision were the sovereign qualities which were always sought.

We find a wider mathematical curiosity in Diophantus who cubed various binomials, especially $(n-1)$. Whether he had a general formula, or multiplied out each time is not clear.

It is a curious fact that the first use, beyond Euclid's, for finding binomial power formulae, was to discover the approximate values of roots. We have a significant remark in the commentary of Eutocius on Archimedes' essay on the measurement of the circle:

"Quo modo adpropinquando radix quadratadati numeri invenienda est, dictum est ab Herone in Metricis a Pappo, Theone, compluribus aliis, qui magnum Syntaxin Claudii Ptolemi interpretati sunt" [1].

This suggests a search in Ptolemy's Syntaxis. I have failed to find the passage. Tannery assures us that Pappus followed the general method of Hero of Alexandria [2]. Hero's method is simplicity itself. If we wish to find an ap-

proximation to $\sqrt{A}$, and a_1 is a first value, a closer one will be

$$a_2 = \frac{1}{2}\left[a_1 + \frac{A}{a_1}\right]. \tag{3}$$

As a matter of fact, this is merely a special case of a famous method of approximating to a simple root of any function, which we associate with the name of Newton, for if a_1 is an approximation to a root of $f(x) = x^2 - A$ a better approximation is

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}.$$

We find something much closer to our familiar method of finding square roots in the work of Theon of Alexandria, who uses our technique of adding to a_1 our correction $(A - a^2)/2a_1$ [3]. Of course it is a question merely of order of procedure, for if we add this correction we have Hero's Formula (3).

We pass to cube roots. Heath says on p. 63, "In no extant Greek writer do we find any description of the method of finding cube roots." If we date Theon about A.D. 390 we have to wait more than 100 years for the Hindu Aryabhata; there are various translations of his Aryabhatiya; I follow that of Datta and Singh, p. 174 [4]:

"Divide the second aghana place by thrice the square of the cube root; subtract from the first aghana place the square of the quotient multiplied by thrice the preceding (cube root) and (subtract) the cube (of the quotient) from the aghana place. The quotient put down at the next place (in the line of the root) gives the root."

I think that this shows clearly enough that Aryabhata was familiar with the binomial formula for a cube. Whether the Hindus had the curiosity to raise binomials to higher powers or not I can not say; a power higher than the third may have appeared to them practically useless. Yet someone must have seen the importance of such matters as may be judged from the following quotation, which is highly significant for the whole purpose of this paper. The writer is Omar Khayyam, and in speaking of a work of his own, now most unfortunately lost, he writes [5]:

"Les Indiens possèdent des méthodes pour trouver les côtés des carrés et des cubes. J'ai composé un ouvrage sur la démonstration de l'exactitude de ces méthodes, et j'ai prouvé qu'elles conduisent, en effet, à l'objet cherché. J'ai en outre augmenté les espèces, c'est à dire j'ai enseigné à trouver les côtés du carré-carre, du cubo-cube, du quadratro cube, à une étendue quelconque, ce qu'on n'avait pas fait précédément. Les démonstrations que j'ai donné à cette occasion, ne sont que des démonstrations arithmétiques, fondées sur les parties arithmétiques des Eléments d'Euclide."

This is an extremely interesting paragraph. Tropfke expresses his opinion in no uncertain terms:

"Die letzte Bemerkung kan man offenbar nur auf Benutzung der binomschen Entwickelung für beliebig hohe Exponenten deuten, wodurch dann Alkhayammi als Entdecker des Binomialtheorems für ganzzählige Exponenten anzusehen wäre" [6].

This seems to me eminently true and important, provided we take it literally. It all depends on "une étendue quelconque." If he could find any root by arithmetical means, he presumably used the binomial theorem, but the only actual roots he mentions are quartic, sextic, and ninth, each of these could be found by repeating the processes he knew for quadratic and cube roots. When we reflect on how inferior was the mathematical notation of his time, I think there is some doubt whether he could really extract, let us say, a seventh root.

A cautious note is sounded in a very recent discussion:

"Man hat die den modernen Mathematiker naheliegende Vermutung ausgesprochen, das Omar Haiyami den binomschen Entwickelung für beliebig hohe Exponenten etwa in der Weise arbeitete wie wir im 16 Jahrhundert bei Apian, Stifel, und andere Mathematiker der Renaissance beobachten" [7].

Luckey does not definitely pronounce on the point in question, but he seems inclined to the view that Omar could only find roots that were based on the quadratic and cubic. I am puzzled by his writing in connection with the quadrato cubic, "Quadratokubus (x^5)." Personally I can not avoid the sentimental hope that he really found the general formula.

2. The arithmetical triangle. We are safe in saying that by the year 1300 at least one capable mathematician was familiar with the binomial expansion for positive integral exponents. Some two hundred years after Omar, there lived in the Flowery Kingdom Chu-Shih-Chieh to whom we are indebted for the interesting diagram

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 1
\end{array}
$$

Mikami comments, "This is indicated as an old calculation but not his invention. . . . This may perhaps have been borrowed from the Arabs in some way" [8]. The horizontal arrangement of the figures suggests strongly that these numbers were found by the expansion of binomials, but of course we have nothing suggesting a proof. Various mathematicians have suggested that the Chinese could expand binomials to quite high powers.

The first writer to give us something really solid is Michael Stifel who published in 1544 his *Arithmetica Integra*. Here we find the diagram

1
2
3 3
4 6
5 10 10
6 15 20
7 21 35 35
8 28 56 70
9 36 84 126 126
10 45 120 210 252

This, of course, can be extracted at once from Chu Shih-Chieh's table, but it is very unlikely that Stifel ever saw the latter. The two significant facts for us are that he was interested in the approximate extraction of roots, and we should like to know the manner in which he explains the construction of his table. In the first column, we have the integers in natural order. Each subsequent column begins two places lower than the preceding one; it starts with the number immediately on its left, and each subsequent number in the column is the sum of the number immediately above and the number to the left of the latter. Now if we write

$$(a + b)^n = (a + b)(a + b)^{n-1}$$

and, if we know the expansion of $(a+b)^{n-1}$, we find the coefficients of the expansion of $(a+b)^n$ by exactly this process. It would seem that Stifel was showing what we should write in modern notation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}. \tag{4}$$

A third form of this figure we owe to Pascal, whose famous triangle appears in the form

1 1 1 1 1 1 1
1 2 3 4 5 6
1 3 6 10 15
1 4 10 20
1 5 15
1 6
1

Probably Pascal was familiar with Stifel's table; he gives the same rule for the

construction of the triangle, as well as some other identities. He points out that the numbers in a N.E. running diagonal are the binomial coefficients, and shows how we find the number of groups of r things taken from n things. Finally in Pascal we have the general rule which we should write [9]

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}. \tag{5}$$

It is important to say that a priority in this has been awarded to others, especially Briggs. Netto writes "Die binomsche Formel findet sich zuerst bei H. Briggs Arithmetica Logarithmica 1620," and Tropfke, that it is on p. 21 of Gellibrand's *Trigonometria Britanica*, a posthumous work of Briggs [10]. This may be. I can only say that I have found no trace. Netto gives no page number and I have seen nothing suggestive of it in the French translation, the only thing available to me; as for Tropfke, all I find on the page in question is a non-triangular form of Pascal's triangle, and there is nothing suggesting Formula (5) in the work.

3. Gregory and Newton. The first writer to approach the binomial expansion of a fractional power was James Gregory, who gave the formula in 1670. His method of approach was curiously indirect, his ostensible desire was to find an antilogarithm. Let us start with two numbers b and d with the logarithms $\log b = e$, $\log (b+d) = e+c$.

To find the number whose log is $(e+a)$,

$$\log b + \frac{a}{c}[\log (b+d) - \log b] = e + a$$

$$e + a = \log\left[b\left(1+\frac{d}{b}\right)^{a/c}\right].$$

Take the two series,

$$b,\ d,\ \frac{d^2}{b},\ \frac{d^3}{b^2}\cdots$$

$$\frac{a}{c},\ \frac{a-c}{2c},\ \frac{a-2c}{3c},\ \frac{a-3c}{4c}\cdots$$

Combine like this

$$b + \frac{a}{c}d + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{d^2}{b} + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{a-2c}{3c}\cdot\frac{d^3}{b^2} = b\left(1+\frac{d}{b}\right)^{a/c}.$$

There is of course no sign of proof [11].

It is time to turn to Sir Isaac Newton to whom we referred somewhat disparagingly in our opening paragraph. The story of his interest in the subject is told at length in a letter to Oldenburg, dated October 1676, and hence six years

after Gregory's letter just mentioned [12]. He tells us that he was interested early in the study of interpolation by Wallis. This admirable mathematician studied curves whose equations were of the type

$$y = (1 - x^2)^{0/2}, \quad y = (1 - x^2)^{1/2}, \quad y = (1 - x^2)^{2/2}, \quad y = (1 - x^2)^{3/2},$$
$$y = (1 - x^2)^{4/2}, \quad y = (1 - x^2)^{5/2}, \quad y = (1 - x^2)^{6/2}.$$

If we take the area of the figure bounded by the positive axes, the curve and the ordinate, we have for the cases of the first, third, fifth and seventh curve

$$x, \quad x - \tfrac{1}{3}x^3, \quad x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5, \quad x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7.$$

How did Wallis discover these formulae, without the aid of integration? He studied quotients of the form

$$\frac{0^p + 1^p + 2^p \cdots n^p}{n^p + n^p + n^p \cdots n^p}$$

and noticed that the limit as n increased indefinitely was asymptotically $1/(p+1)$, this worked out at least in several instances [13]. We pass easily from this to finding the area under the curve $y=x^p$. We divide into n parts the segment from the origin to $x=n\Delta x$ on the axis, the points of division being 0, Δx, $2\Delta x$, $\cdots$, $n\Delta x$. We erect rectangles on these, one upper vertex being on the curve. The areas will be

$$\Delta x^p \cdot \Delta x, (2\Delta x)^p \cdot \Delta x, (3\Delta x)^p \cdot \Delta x, \cdots, (n\Delta x)^p \cdot \Delta x.$$

The sum will be

$$(\Delta x)^{p+1}(1^p + 2^p + 3^p + \cdots + n^p).$$

Now

$$\lim_{n\to\infty} \frac{(0^p + 1^p + \cdots + n^p)}{(n+1)n^p} = \frac{1}{p+1};$$

$$\lim_{n\to\infty} (\Delta x)^{p+1}[0^p + 1^p + 2^p + \cdots + n^p] = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{(n\Delta x)^{p+1}}{p+1} = \frac{x^{p+1}}{p+1}.$$

This gives the fundamental formula

$$\int x^p dx = \frac{x^{p+1}}{p+1}. \tag{6}$$

Naturally Wallis did not set things up in anything like this form but such is the essence of his reasoning. Moreover, an equivalent formula for quite a number of cases had been proved by Cavalieri and others [14]. Wallis knew also that the integral of a sum is the sum of the corresponding integrals, so there is no real mystery about his discovery of these areas.

Let us return to Newton. He wished to find the areas under the other curves.

beginning with the second which is the circle $y=(1-x^2)^{1/2}$. Newton notes that in all of Wallis' cases the first term is x and the denominators are 1, 3, 4, 7, $\cdots$. They cause no trouble. They come in through the integration, not through the expansion. The second terms are:

$$-\tfrac{0}{3},\ -\tfrac{1}{3},\ -\tfrac{2}{3},\ -\tfrac{3}{3}\cdots.$$

Now $(1-x^2)^{(k+1)/2}$ is a mean proportional between $(1-x^2)^{k/2}$ and $(1-x^2)^{(k+2)/2}$, and this gives the first numerator in cases 1, 3, 5, $\cdots$. He guesses that this averaging process works in every case, and the other expansions should begin

$$x-\frac{\frac{1}{2}x^3}{3},\quad x-\frac{\frac{3}{2}}{3}x^3,\quad x-\frac{\frac{5}{2}}{3}x^3\cdots.$$

Suppose, then, the expansion begins $z-(m/3)x^3$. How shall the subsequent terms be found? Let us follow his own words [**15**], "Quaerebam itaque quomodo in his seriebus ex datis duobus primus figuris reliquae derivari present. Et inveni quod posita secunda figura, reliquae producerentur per continuarum multiplicationem terminorum hujus serie"

$$\frac{m-0}{\quad}\times\frac{m-1}{\quad}\times\frac{m-2}{\quad}\times\frac{m-3}{\quad}\times\cdots.$$

The essential word here certainly is "inveni." Did Newton work this out for himself or, more likely, did he follow Pascal's and Stifel's formula which holds in the integral case, and guess that it was always correct, and then work out some cases? We shall never know the answer, and on this, largely, I think, depends the amount of credit which he should receive. All that we surely know is that he wrote out

$$(1-x^2)^{1/2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{3/2}=1-\frac{3x^2}{2}-\frac{3x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{1/3}=1-\frac{x^2}{3}-\frac{x^4}{9}-\frac{5x^6}{81}\cdots.$$

He squared or cubed the series and reached $(1-x^2)$. In the first case he found the square root by the usual method and reached the series. I cannot see that he did more than this, in which case, brilliant as was his genius in other matters, I do not think he deserves extraordinary credit for his contribution to the binomial theorem.

4. Attempts at proof. What shall we now say about demonstrations of the theorem; have any of these writers really proved it, or have they merely followed the example of all beginners, showing merely that no other solutions are

possible? We have seen that Omar Khayyam had doubts on the point, he says "J'ai prouvé qu'elles conduisent en effet à l'objet cherché." Newton's verifications, as far as they go, show that the series are equal to binomials. Pascal in another connection used mathematical induction. He was one of the first users, but he did not prove our theorem in this way. How did anyone know *a priori* that any non-integral power of a binomial was actually equal to a certain convergent series?

In 1742 there appeared an article by Giovanni Salvemini who lived in Castiglione, for which insufficient reason he was frequently referred to as De Castillon. In the *Philosophical Transactions*, vol. 42, 1742–3, we find his article. He points out that everyone knows Newton's formula, but no one, as far as he knows, has proved it. He distinguishes three cases (a) a positive integral exponent, (b) a positive fractional exponent, (c) a negative exponent. The first case he handles by a method still used today. Let us replace $(p+q)^n$ by the product $(p+q_1)(p+q_2)\cdots(p+q_n)$. The numerical coefficient of the term of the rth order in the q's will be the number of combinations of n things taken r at a time, namely $\binom{n}{r}$. Then we set all the q_i's equal to q. When it comes to expanding $(p+q)^{r/n}$, we are safe in taking the first exponent as r/n, for that is the case when $q=0$; thus

$$(p+q)^{r/n} = Ap^{r/n} + Bp^{r/n-1}q + Cp^{r/n-2}q^2 + \cdots$$
$$(p+q)^r = p^r(A + Bp^{-1}q + Cp^{-2}q^2 + \cdots)^n.$$

He knows how to expand a binomial to any positive integral power, and blithely expects that he can do the same thing with a convergent power series, treating it as a binomial. In expanding, the new coefficients kindly come in one at a time, so that we have

$$1 = A^n; \qquad r = nA^{n-1}B; \qquad \frac{r(r-1)}{1\cdot 2} = nA^{n-1}C + \frac{n(n-1)}{1-2}A^{n-2}B^2$$

$$1 = A; \qquad \frac{r}{n} = B; \qquad \frac{\frac{r}{n}\left(\frac{r}{n}-1\right)}{1\cdot 2} = C.$$

The negative expansion comes by taking the reciprocal of the positive one.

A much quicker method was devised by a far abler mathematician, Colin Maclaurin. Here is what he writes on pp. 607–8 of vol. 2 of his *Fluxions*, Edinburgh, 1742:

"Let it be required to find $\overline{1+x}^n$ where n may represent any integer number or fraction, positive or negative. It is evident from what is shown in the common algebra concerning powers and their roots that the first term of any power of $(1+x)$ is 1, and the subsequent terms involve x, x^2, x^3, x^4, $\cdots$ with invariable coefficients. We suppose therefore

$$\overline{1+x}^n = 1 + Ax + Bx^2 + Cx^3 + \cdots$$

represents the general formula. By taking the fluxions on both sides

$$n\dot{x}\,\overline{1+x}^{n-1} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x}.$$

This is an identity, hence if we take $x=0$ (or because the first term of $\overline{1+x}^n$ must be 1) we must have $A=n$." The other coefficients are quickly found by similar processes and further differentiation.

This demonstration was not essentially new, it appeared five years earlier in the work of Colson [16]. The reasoning is less clear as he uses the the same letter to mean two different things; he writes on succeeding lines $y=\overline{a+x}\,|^m$ and $y=a^m$. However he comes out all right in the end. But he makes on p. 310 an important remark which seems to have escaped Maclaurin:

"Indeed it can hardly be said that this or any other that is developed from the Method of Fluxions is a strict investigation of this Theorem. Because the Method itself is originally derived from the method of raising Powers, at least integral Powers, and presupposes the knowledge of Unciae or numerical coefficients."

Exactly this same difficulty occurred somewhat later to Euler, who gave two other demonstrations, of which I reproduce the second [17]. We start with the equations

$$\begin{aligned}(1+x)^n &= 1 + Ax + Bx^2 + Cx^3 + \cdots \\ (1+x)^{n+1} &= 1 + A'x + B'x^2 + C'x^3 + \cdots \\ &= (1+x)(1+x)^n.\end{aligned}$$

Suppose n is an integer. When $n \leqq 0$, all coefficients vanish; when $n \leqq 1$ all after A; when $n \leqq 2$ all after B; and so on. Let us write

$$\begin{aligned}A &= \alpha n, \qquad B = \beta n(n-1), \qquad C = \gamma n(n-1)(n-2), \cdots \\ (1+x)^{n+1} &= 1 + \alpha(n+1)x + \beta(n+1)nx^2 + \cdots \\ &= (1+x)(1 + \alpha nx + \beta n(n-1)x^2 + \cdots .\end{aligned}$$

Subtracting

$$0 \equiv (\alpha - 1)x + (2\beta n - \alpha n)x^2 + \cdots .$$

Dividing out x, and setting $x=0$, we have

$$\alpha = 1, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{1}{2\cdot 3},$$

and so on. Euler concludes, "Prorsus superfluum foret hos casus ulterius prosequi, cum jam luce meridianam clarius apparat pro singulis litteris sequentibus eosdem plane valores necessario prodire debere, quos evolutio New-

tonianae docuit, atque haec demonstratio naturae rei tam apprime accomodato videtur, ut illi etiam in primis Analyseos elementis denegeri nequeat. Quin etiam universum ratiocinium qui hic usi sumus, unam vim retinet, etiamso adeo n ut imaginarius spectaretur" [18].

I must confess that I am not much impressed by this proof. It has the important advantage of being equally applicable to all values of the exponent but the best reason for the assumption as to the form of the coefficients is that it is correct in the integral case; the statement that something is "luce meridiana clarius" is not the same thing as a mathematical demonstration.

5. Convergence. There remains the important problem of the convergence of the series. The early writers were more or less aware of the existence of this question but were unable to handle it completely. The honor for doing this goes to Niels Hendik Abel. His contribution is much too long to be repeated here [19].

LEMMA (1) *Let* $\rho_1\rho_2\rho_3\cdots\rho_n$ *be a series of positive quantities such that* $\lim \rho_{m+1}/\rho_m=\alpha>1$ *and* ϵ_m *be quantities whose absolute values do not approach* 0 *as a limit as* m *increases without limit, then the series* $\epsilon_0\rho_0+\epsilon_1\rho_1+\epsilon_2\rho_2\cdots$ *is divergent.*

We see in fact that regardless of how great m may be, the set

$$\epsilon_m\rho_m+\epsilon_{m+1}\rho_{m+1}+\cdots+\epsilon_n\rho_n$$

which may contain positive or negative terms, will not approach 0 as a limit.

LEMMA (2) *If in the above series* $\alpha<1$, $|\epsilon_m|<A$, *the series is convergent*

We see, in fact that the absolute value of the set is less than $\rho_m A 1/(1-x)$, but $\lim \rho_m=0$.

Now Abel makes a trigonometric development by the use of De Moivres' theorem.

Let

$$\phi(m)=1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots.$$

Let

$$x=a+bi, \qquad m=k+k'i; \qquad \phi(m)=p+qi;$$

$$\sqrt{a^2+b^2}=\alpha; \qquad x=\alpha[\cos\phi+i\sin\phi]$$

$$\frac{m-\nu+1}{\nu}=\delta\nu[\cos\gamma\nu+i\sin\gamma\nu]$$

$$\binom{m}{\mu}=\alpha^\mu\delta_1\delta_2\cdots\delta_\mu[\cos(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu)$$

$$+i\sin(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu)].$$

Abel here treats the real and imaginary parts separately, the question of convergence depends, as above, on whether α is greater than or less than unity. The case where it is equal to one, he treats at length, separately. One has the feeling that the last word has been said.

Yet that again is not the case. We have a variety of proofs that, if any power of a binomial can be expressed as a series of positive integral powers, this is the series. But why should such a series exist *a priori?* Omar sensed this difficulty. De Castillon met it, in the case of rational powers of the binomial, and, if his method were strengthened by showing that the algebraic operations with an infinite series were legitimate, and, that an extension by continuity considerations from the rational to the irrational is permissible, we might find the whole here. In a few cases Newton showed that the series which he developed did what they were supposed to do. Perhaps the easiest thing to do would be to find a general proof independent of the binomial theorem that $\dot{x}^n = nx^{n-1}\dot{x}$; then, with the aid of Taylor's Theorem with remainder, give MacLaurin's proof for the binomial case. What a long distance from Euclid II, 4!

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SOME APPLICATIONS OF EXTENDED ANALYTIC GEOMETRY

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1. Introduction. This paper deals with techniques developed for the case of three axes in extended analytic geometry,* together with some of the applications in number theory which are suggested thereby. While the subject has intrigued the writer on its own merits, it is only natural that readers will be more inclined to give it their attention if they can be shown that it solves problems in other fields.

2. The basic system. On the 3-axes plane the positive halves of the x, y, and z axes radiate from the origin at standard position angles of 0°, 60°, and 120°.† The three coördinates of a point designate directed segments which are parallel to the respective axes and are applied in vector sequence, starting from the origin. It is evident that the order of application is immaterial, and that one set of coördinates designates one and only one point.

We are now prepared to state, by way of preview, some of the reasons why the case of three axes should receive special attention in any study of the proposed system: (1) It is, of course, the simplest case of the n-axes plane when $n>2$. (2) The fact that it constitutes a plane analogue for solid analytic geometry invites comparison of the two subjects at many points, and suggests studies in the new field paralleling those in the old. (3) The application to number theory and particularly to Diophantine equations, is facilitated by a combination of four happy circumstances: (a) The equation of a line or a curve is a single one rather than a pair, as in the solid case; (b) the equation of a straight line through points with integral coördinates may be written with integral coefficients (this fails when $n>3$ unless the axes are spaces unequally, as will be explained later); (c) the points designated by all-integral coördinates lie in a simple pattern at the vertices of equilateral triangles filling the plane; (d) the "one degree of freedom" in choosing the coördinates of a given point is very helpful, as we shall see.

As a matter of fact, the simplicity of results for the 3-axes plane seems less surprising when we note that, from one point of view, its loci are actually the projections on a plane of the surfaces of solid analytic geometry. The plane may be considered to pass through the origin and to be perpendicular to the line segment from (0, 0, 0) to (−1, 1, −1), if we use the right-handed or clockwise XYZ system. The observer's eye is "at infinity" in the direction indicated by the segment. But though this background consideration helps one at times to predict the nature of certain loci, and even gives interesting significance, from the standpoint of projection, to changes in the spacing of the three axes, it is neither essential nor desirable as a habitual way of viewing the system. Certainly the point of view which regards loci on the n-axes plane, when $n>3$, as

* See An Analytic Geometry for N Variables, by R. S. Underwood, this MONTHLY, vol. 52, May, 1945. Henceforth this will be referred to as Paper I.

† For the relevant figures, see Paper I.

"projections of configurations in n-space," is far-fetched as well as, probably, barren and unprofitable.

3. The basic equations. The coördinates x_i of a point on the n-axes plane are related to the rectangular coördinates (X, Y) with references to a superimposed pair of axes by the equations,*

$$(1) \qquad X = \sum_{i=1}^{n} x_i \cos \theta_i; \qquad Y = \sum_{i=1}^{n} x_i \sin \theta_i; \qquad \theta_i = (i-1)\pi/n.$$

In the case of three axes, θ_1, θ_2, and θ_3 are respectively 0°, 60°, and 120°, and the fundamental equations, after simplification, take the form

$$(2) \qquad 2x + y - z = 2X; \qquad y + z = \frac{2}{\sqrt{3}} Y.$$

Throughout this paper the letters x, y, and z denote the coördinates of a general point on the 3-axes plane, while X and Y are the corresponding rectangular coördinates on the superimposed 2-axes plane.

4. The straight line. Theorem 2 of Paper I may be restated for the case of three axes as follows:

THEOREM 1. *The locus of the equation*

$$(3) \qquad Ax + By + Cz = D$$

is a straight line when $B = A + C$; otherwise it consists of all the points on the 3-axes plane.

COROLLARY. *The general equation of a straight line on the 3-axes plane may be written in the form*

$$(4) \qquad Ax + By + (B - A)z = C.$$

The proof of the theorem may be obtained by solving (3) simultaneously with the two equations in (2). When the determinant of the coefficients of x, y, and z is not zero, real values for these letters are found to correspond with each real pair X and Y; otherwise the equations are inconsistent except for points (X, Y) on a line.

At this point it may be noted parenthetically that when the analogous method is tried on the equation of first degree in four variables, it is found that if θ_1, θ_2, θ_3, and θ_4 are 0°, 45°, 90°, and 135°, respectively, the general equation of the straight line on the 4-axes plane becomes

$$Ax + By + (\sqrt{2}\, B - A)z + (B - \sqrt{2}\, A)u = C,$$

so that the possibility of integral coefficients is barred. If, however, the four

* Paper I, equations (7), p. 256.

angles are made respectively 0°, arctan (4/3), 90°, and 180°−arctan (4/3), the equation becomes

$$Ax + By + \frac{(5B - 3A)}{4} z + \frac{(5B - 6A)}{5} u = C.$$

This will explain the parenthetical statement following item (b) in the third paragraph of this paper.

5. The equation of a line through two given points. By a convenient extension of the familiar definition of a line's slope which will apply generally when $n>2$, we may define the slope* m of the line (4) as $-A/B$, including zero and infinite slopes. Geometrically, we may fix the slope of (4), or of any line on the n-axes plane, as "minus the ratio of the y-intercept to the x-intercept," where x and y are the first two variables.

The equation of the line through the points (a, b, c) and (d, e, f) is

$$(e + f - b - c)x + (a + f - c - d)y + (a + b - d - e)z = ae + af + bf - bd - cd - ce, \tag{5}$$

as may be verified by substitution. It follows that, for this line,

$$m = \frac{(b + c) - (e + f)}{(a + f) - (d + e)}. \tag{6}$$

Given two points, or the slope and one point on a line, its equation may be written almost immediately. For example, for the line through (2, −3, 4) and (5, 0, −1), $m=-\frac{1}{4}$, whence the left side of the equation is $x+4y+3z$ and the right side is 2, as found and checked by using the coördinates of both points.

6. Generalized coördinates. From (2) it follows that if (a, b, c) are a given set of coördinates of a point $P(X, Y)$, then any set of coördinates of P, say x, y, and z, are related to a, b, and c as follows;

$$2x + y - z = 2a + b - c; \qquad y + z = b + c. \tag{7}$$

Now if we eliminate y and z successively from (7) we get $y=a+b-x$ and $z=c-a+x$. Therefore, the coördinates

$$(x, a + b - x, c - a + x), \tag{8}$$

where x is arbitrary, may be called the generalized coördinates, (abbreviated as G.C.) of the point (a, b, c). The G.C. prove to be extremely useful in the theory, since they allow a choice of coördinates of the given point which satisfy an extra condition. To illustrate, the special coördinates of the point (−2, 3, 4) which satisfy the equation $3x+4y+5z=6$ are (−7, 8, −1), which we find by substituting the G.C., $(x, 1-x, 6+x)$, in the equation and solving for x.

* This is consistent with the statement in Paper I (p. 255) that $m_{xy}=\partial y/\partial x$, $m_{yz}=\partial z/\partial y$, etc. Here $m=m_{xy}$.

7. An application to Diophantine equations. Suppose we seek simultaneous integral solutions of the equations

$$5x - 6y - 7z = 4 \tag{9}$$

$$7x + 11y - 3z = 5. \tag{10}$$

Without benefit of illumination from geometry we might try eliminating letters, getting successively $34x+95y=23$, $97x+95y=74$, and $97x+34z=3$. From the first of these equations, $x=(23-95y)/34$ so that we must solve the congruence $95y\equiv 23 \pmod{34}$, or some equally tedious one, to get integral solutions.

Consider next the geometric method. Upon multiplying (10) by k and adding to (9), we get

$$(5 + 7k)x + (-6 + 11k)y + (-7 - 3k)z = 4 + 5k, \tag{11}$$

In solid analytic geometry we would interpret (11) as the equation of a family of planes whose members pass through the line of intersection of the planes (9) and (10). On the 3-axes plane the interpretation is essentially the same; but in this case we can find the equation of the common line simply by fixing k so that

$$-6 + 11k = (5 + 7k) + (-7 - 3k). \tag{12}$$

This yields $k=4/7$, whence (11) becomes

$$63x + 2y - 61z = 48. \tag{13}$$

We may call (13) the equation of the line of intersection of the loci of (9) and (10). Only points on this line have coördinates which satisfy (9) and (10) simultaneously; and furthermore, every point on the line has exactly one such set of coördinates.

The line (13) is "infinitely many layers deep" in the sense that all coördinates of every point on it satisfy its equation. (This statement is not true for a typical locus on the 3-axes plane such as, for example, the plane $x+y+z=1$ and the area $x^2+y^2+z^2=1$, which are respectively one and two layers deep.) It follows that we may assign a value at will to one of the letters, thus using the one degree of freedom, and then "scan" the line with the remaining two coördinates, reaching every point on it. Letting $x=0$ by preference we find, upon solving the simple congruence,

$$61z \equiv -48 \pmod{2}, \tag{14}$$

that $z=2n$ and $y=24+61n$. Thus all of the lattice points on the line (or points whose coördinates are all integers) are designated by the coördinates $(0, 24+61n, 2n)$, where n is an integer. To find the special coördinates of these points which satisfy (9) and (10) simultaneously, we substitute their G.C. $(x, 24+61n-x, 2n+x)$ in (9) [and also in (10) for checking purposes]. This yields $x=37+95n$, and hence, from the G.C., one form of the complete integral solution of

(9) and (10) is

$$x = 37 + 95n, \qquad y = -13 - 34n, \qquad z = 37 + 97n. \tag{15}$$

In comparing the merits of the method used above with the purely algebraic one by which a letter is eliminated initially, it should be emphasized that neither method is *per se* the "more efficient" one. In both cases it is sometimes necessary to solve a second congruence to complete the solution, and each method excels on occasion, when the problem is chosen to that end. However, there are obvious simplifications of the geometry-guided method in many special situations, as when the locus of one of the original equations is itself a straight line, or can be made a line by the interchange of two letters, or again when both loci are lines. In such cases the geometric background casts helpful light on the various situations.

Nevertheless, the full potentialities of the methods associated with the 3-axes plane are not seen as long as we neglect equations of the second or higher degree. Evidently a useful result would be a simple method for obtaining the equation of the curve of intersection of two loci in general. Such a method is found in the next article.

8. Equations of curves. Normally the locus of a single equation is an area, or, from another point of view, a family of curves which collectively cover some determinable area on the 3-axes plane. But sometimes the loci are single curves which are analogous to the cylindrical surfaces of solid analytic geometry "seen on edge." In fact, we may translate the equation

$$f(X, Y) = 0 \tag{16}$$

into the equation in three variables of the designated curve simply by replacing X and Y by the expressions for them in (2). For example, the equation of the circle $X^2+Y^2=a^2$ becomes, on the 3-axes plane,

$$x^2 + y^2 + z^2 + xy - xz + yz = a^2. \tag{17}$$

By the *curve of intersection* of the loci of two equations we mean the curve containing all of the points, and only the points, which have coördinates that satisfy both equations simultaneously. The points have also, of course, other coördinates which satisfy the equation of the curve but not of the intersecting loci. In fact, the curve of intersection, like (13), is infinitely many layers deep, and in particular its equation is satisfied by the members of three unique sets $(0, y, z)$, $(x, 0, z)$, and $(x, y, 0)$. Thus when one variable is made zero, the remaining two variables and two axes are sufficient for plotting the curve.

THEOREM 2. *All points on the curve of intersection of the loci of $f(x, y, z)=0$ and $g(x, y, z)=0$ are also on the curve obtained by replacing x, y, and z in those equations by x', $x+y-x'$, and $z-x+x'$, respectively, and then eliminating x'.*

In fact, the curve thus obtained usually is the curve of intersection, though

it may contain additional points.* For convenience we shall call it the *plus curve of intersection.*

The proof of the theorem follows. If we replace y and z by $Y/\sqrt{3}+X-x$ and $Y/\sqrt{3}-X+x$ respectively [from (2)] and then eliminate x, we get a relation between X and Y, say $F(X, Y)=0$, which must hold for all coördinates (x, y, z) of points on the curve of intersection. Since $F(X, Y)=0$ itself designates a curve, real or imaginary, this curve must coincide with or include the curve of intersection. To change back to three variables we replace $Y/\sqrt{3}+X$ and $Y/\sqrt{3}-X$ by $x+y$ and $z-x$ respectively. The complete operation is telescoped in the theorem.

For an application, we shall find integral solutions of

$$x^2 + xy + ay^n = z^2 \tag{18}$$

by solving it simultaneously with

$$x + y - z = m. \tag{19}$$

Through the substitutions of Theorem 2 in (19), we get $x'=2x+y-z-m$. Using this value of x' in the expressions for y and z, ($x+y-x'$ and $z-x+x'$), we find that if we replace x, y, and z in (18) by $2x+y-z-m$, $z-x+m$, and $x+y-m$, respectively, we get the equation of the plus curve of intersection of (18) and (19). More simply, we may replace x, y, and z by $y-z-m$, $z+m$, and $y-m$, respectively, getting the same curve, though now, with $x=0$, it is not infinitely many layers deep. This yields

$$y = \frac{z^2 + 2mz - (m+z)^2 + a\,(m+z)^n}{z - m},$$

so that, if $z=m+1$, y is the integer $a(2m+1)^n-m^2$. We then substitute the G.C. of the points $[0, a(2m+1)^n-m^2, m+1]$ in (19), getting $x=a(2m+1)^n-(m+1)^2$. Using the G.C., we then have the following solution of (18), with m arbitrary:

$$x = a(2m + 1)^n - (m + 1)^2;\ y = 2m + 1;\ z = a(2m + 1)^n - (m^2 + m). \tag{20}$$

The accomplishment here is less than it will appear to be at first, since integral solutions for (18) might be obtained by replacing z by $x+u$ and solving for x. However, the fact that a true algebraic result may often be obtained from transformations other than those suggested by the geometric background is to be expected. At times the geometric method leads to an easy solution which in turn suggests an even simpler non-geometric approach. Though this cannot be said to refute the usefulness of the geometry, it does indicate the nature of the game which the writer has found it necessary to play in his search for a conquest with the new tools which he could not (at least without considerable difficulty) duplicate otherwise, even with the aid of "hindsight." We shall close with three theorems which seem to meet this requirement. It is to be understood that all "solutions" are restricted to sets of integers.

* For example, the plus curve of intersection of the loci of $x^3+y+z+1=0$ and $x^3+1=0$ is $y+z=0$ (the z-axis), though $x^3+1=0$ has no locus and hence there is no curve of intersection.

THEOREM 3. *If*

$$f(u, v) = 0, \qquad (u = 2x + y - z, v = y + z) \tag{21}$$

considered as an equation in x, y, and z, is solvable when $x=0$ (or $y=0$, or $z=0$), it is also solvable when either of the other two letters is zero, and furthermore, it has infinitely many solutions; but if the first premise fails, (21) *has no solutions.*

THEOREM 4. *The equations*

$$x^2 + y^2 = xz, \quad \textit{and} \tag{22}$$

$$z^2 + xy + yz = n \tag{23}$$

have no common solution when $n = 2, 5, 6, 8, \cdots$; that is, when n is an integer not expressible in the form a^2+ab+b^2, with a and b arbitrary.

THEOREM 5. *The equations*

$$x^2 + y^2 + z^2 + xy - xz + yz = n \quad (n \neq 0), \quad \textit{and} \tag{24}$$

$$x^2 + y^2 - z^2 + xy + xz - yz = 0 \tag{25}$$

have no common solution.

Proofs. (Theorem 3.) Equation (21) is that of the curve $f(2X, 2Y/\sqrt{3}) = 0$. If this curve passes through a lattice point with one set of coördinates (a, b, c), its equation will be satisfied by all coördinates of the point, in accord with previous statements and also as shown by replacing (a, b, c) by the G.C., $(x, a+b-x, c-a+x)$. It will therefore still be satisfied when x or y or z is replaced by zero. But if the curve "misses" all lattice points, its equation cannot have solutions.

(Theorem 4.) The equation

$$(x^2 + y^2 - xz) + (z^2 + xy + yz - n) = 0 \tag{26}$$

will evidently be satisfied by all common solutions of (22) and (23). [Incidentally (26) is the equation of the curve of intersection, though it is not here obtained by use of Theorem 2.] But (26) is the circle $X^2+Y^2=n$, and hence common solutions of (22) and (23) can occur only when n has the form a^2+ab+b^2, since otherwise, by Theorem 3, (26) has no solutions. [Actually (22) and (23) have common solutions, namely $(-4m, 4m, -8m)$, when $n=16m^2$; and do not have common solutions when $n=a^2+ab+b^2$ with $a>0$ and $b>0$.]

Theorem 4 may be generalized in various ways.

(Theorem 5.) Adding the terms of (24) and (25), we get $n=2(x^2+xy+y^2)$. But since, as in the preceding proof, n must have the form a^2+ab+b^2 to yield solutions for (24), and since n as found above is even, it is necessary that $a=2A$ and $b=2B$, whence $n=2(x^2+xy+y^2)=4(A^2+AB+B^2)$. This again requires that x and y, and then A and B, both be even, and so on. The argument by "infinite descent" leads to the conclusion of the theorem.

NOTES ON QUARTIC CURVES

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1. Introduction. In these notes certain properties of the Cartesian graph of the quartic function

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

are considered. Some of these, though well known, are linked to other relationships which have turned up in classwork. Thus certain characteristics are pointed out which may assist in the sketching, and add to the understanding of quartic curves.

For convenience, a translation of axes will be applied to remove the term in x^3. Also, a change in the scale of the y-coördinates will, without loss of generality, allow the quartic to be taken in the form

$$y = f(x) = x^4 + px^2 + qx + s. \tag{1}$$

From the expression in (1) it will be seen that of all the lines which meet the quartic curve it is only those of slope q which, upon solving for the points of intersection, will give quadratic equations in x^2. This means that lines of slope q can be used to fix points on the quartic which can be paired so that their abscissas are numerically equal but opposite in sign. In other words the quartic curve (1) is skew-symmetric with respect to the y-axis along lines of slope q. Thus the y-axis is the locus of the midpoint of all chords of slope q, when these chords are drawn to points properly paired. Furthermore, since the equation of the tangent to the quartic at the point (x_1, y_1) can be written in the form

$$y = (4x_1^3 + 2px_1 + q)x - 3x_1^4 - px_1^2 + s, \tag{2}$$

it is seen that the y-intercept of the tangent is unchanged when $-x_1$ is substituted for x_1. It follows that the two tangents drawn to the quartic from the properly paired endpoints of any chord of slope q will meet on the y-axis. On account of these properties this line may therefore be considered as a pseudo-diameter of the quartic.

2. Further significance of the coefficients. In addition to this significance attached to the coefficient q it is well to note a relationship which exists between two quartic curves whose equations of form (1) differ only in the sign of q. Denote such two quartics by $y=f(x, q_1)$ and $y=f(x, -q_1)$. Then by (1) there exists the identity

$$f(x, q_1) \equiv f(-x, -q_1),$$

which shows that the two quartic curves are symmetrical with respect to the y-axis. Therefore sufficient information in regard to curves of all quartics like (1) can be gathered by limiting q to positive values. Each quartic equation of form (1) for which $q=0$ can be represented by a curve which is symmetrical

with respect to the y-axis.

The significance of the parts played by the constants q and s in equation (1) should now be evident. Variation of the parameter s brings about a translation, while changes in the parameter q affects the skew-symmetry of the quartic with respect to the y-axis. But when the coefficient p is allowed to change there will result such variations in the graphs that it seems best to divide the discussion into two parts according as the quartics have $p \geqq 0$, or $p < 0$. The latter group will present more points of interest and will be considered first. Therefore in the following discussion, unless otherwise stated, the coefficient p will represent a negative number.

There are three lines of slope q that may be of use in making a sketch of a quartic curve of this second group. The first is the tangent-secant line

$$y = qx + s \tag{3}$$

which is tangent to the graph of (1) at the point $(0, s)$. It will also cut the quartic at the two points where $x^2 = -p$.

The second is the double-tangent line

$$y = qx + s - \frac{p^2}{4} \tag{4}$$

which is tangent to (1) at the two points where $2x^2 = -p$.

The third line is the secant through the two points of inflection. These points exist only when $p < 0$. This line, the inflection secant, may be found by dividing the quartic function in (1) by $1/12 f''(x)$ until the remainder $r(x)$ is of first degree or lower. Let the quotient be $q(x)$. It is then true that

$$f(x) = \frac{1}{12} f''(x) \cdot q(x) + r(x). \tag{5}$$

The function $r(x)$ will equal the function $f(x)$ for each of the two values of x for which the second derivative $f''(x)$ vanishes. Therefore, $y = r(x)$ is the equation of the line through the two points of inflection. It turns out that

$$\frac{1}{12} f''(x) = x^2 + \frac{p}{6}, \qquad q(x) = x^2 + \frac{5p}{6},$$

and the equation of the inflection secant is

$$y = r(x) = qx + s - \frac{5p^2}{36}. \tag{6}$$

The two points of inflection P_1, P_2 have $x^2 = -p/6$. It will be seen also that the inflection secant (6) will cut the quartic again in two points Q_1 and Q_2, the abscissas of which are found from the equation $q(x) = x^2 + 5p/6 = 0$. For the two functions $f(x)$ and $r(x)$ are equal for each of the two zeros of the function $q(x)$.

3. The quartic and its inflection secant. At this point it is well to note certain properties or relations between the quartic (1) and its inflection secant (6). Assume that the four points Q_1, P_1, P_2, Q_2 on the inflection secant are taken in this order from left to right. Since the diameter $x=0$ bisects all chords of slope q, it follows that the midpoints of the segments Q_1Q_2 and P_1P_2 coincide, whence

$$Q_1P_1 = P_2Q_2. \tag{7}$$

The quartic curve cuts off two equal segments on the inflection secant.

Since the abscissas of these four points are known, then by means of the x-projections of the line segments it will be seen that

$$P_1Q_2 : P_1P_2 = (\sqrt{5}+1):2, \quad \text{and} \quad P_1P_2 : P_2Q_2 = 2:(\sqrt{5}-1).$$

It is true therefore that

$$P_1Q_2 : P_1P_2 = P_1P_2 : P_2Q_2. \tag{8}$$

This shows that the line segment P_1Q_2 is divided by the point P_2 in the "golden section." A similar situation exists for the line segment Q_1P_2 and the point P_1.

From plane geometry it is known that for a circle of radius r the side of the regular inscribed decagon c_{10} is $(r/2)(\sqrt{5}-1)$. It follows from the preceding statement that the three segments Q_1P_1, P_1P_2, and P_2Q_2 are so related that if

$$r = P_1P_2, \quad \text{then} \quad c_{10} = Q_1P_1 = P_2Q_2. \tag{9}$$

A sketch will show that the inflection secant and the quartic curve enclose three distinct areas. A student, E. W. Hofler, has pointed out and proved by integration that two of these areas are equal, and that the largest area of the three is equal to the sum of the other two, (problem E817, this MONTHLY, May, 1948). It is also true for all values of p that the area enclosed by the quartic (1) and any line of slope q is divided by the y-axis into two parts of equal area.

4. Sketching the curve, bend points. The three parallel lines (3), (4) and (6) will prove to be of use in sketching a quartic curve of this group. For notation let T_1 be the point of contact $(0, s)$ of the tangent-secant line. Let S_1 and S_2 (left to right) be the two points in which this line (3) cuts the curve. For the double tangent (4) let T_2 and T_3 be the two points of contact. The abscissas of these points are known. On the inflection secant the four points Q_1, P_1, P_2 and Q_2 have already been used. If three real bend points exist and if q is positive, let B_1 be the bend point, a maximum point. It is located on the curve a little to the right of T_1. The other two bend points B_2 and B_3 are points of minima. They will be near, but to the left of the points T_2 and T_3. This description covers what is usually the part of the quartic curve which is of greatest interest. This part is somewhat like a W with curved turns. Changes in the parameter p cause changes in the size of the W-part. This part lies between the tangent-secant line (3) and the double-tangent line (4). Their distance apart measured in the y-

direction of the y-axis is always $p^2/4$ units. The spread of the W-part is noted from the points S_1 and S_2 on line (3) which may be considered as the upper extremities of the W. Their abscissas are given by $x^2=-p$. The spread of the lower part of the W is given by the points T_2 and T_3 which lie on the line (4) for which the abscissas are $2x^2=-p$. These lines and points will locate the W-part and will show how its size varies as the parameter p moves over the range $p<0$.

Next, consider the conditions for the existence of real points of inflection and three real bend points. The first and second derivatives of (1) are

$$f'(x) = 4x^3 + 2px + q, \quad \text{and} \quad f''(x) = 12x^2 + 2p. \tag{10}$$

From the second derivative it is seen that the quartic will have none or two points of inflection according as $p \geqq 0$, or p is negative.

Turning to the bend points, the condition for the existence of three real bend points is that the first derivative function, or its equivalent the cubic function

$$\frac{1}{4}f'(x) = x^3 + \frac{p}{2}x + \frac{q}{4},$$

will have three distinct zeros. From the theory of cubics it is known that this requires that the following inequality holds

$$\left(\frac{p}{6}\right)^3 + \left(\frac{q}{8}\right)^2 < 0.$$

Expressing q as a function of p, also solving for p in terms of q, will give the two equivalent statements

$$q^2 < -\frac{8}{27}p^3, \quad \text{or} \quad p < -\frac{3}{2}q^{2/3}. \tag{11}$$

It is only when the relations in (11) hold that the quartic curve (1) will have three bend points.

It may be of interest to note that through the three real bend points of a quartic it is possible to pass a parabola of the form $y=ax^2+bx+c$. Its equation can be found by dividing the quartic function $y=f(x)$ in (1) by the function $1/4\,f'(x)$. Let $Q(x)$ be the quotient and $R(x)$ the remainder. It is true then that

$$f(x) = Q(x)\cdot\frac{1}{4}f'(x) + R(x).$$

Assume that the first derivative function $f'(x)$ vanishes for each of the three distinct values $x=r_1, r_2, r_3$. It is seen then that the two functions $f(x)$ and $R(x)$ are equal for each of the three values $x=r_1, r_2, r_3$. Furthermore, the zero of the function $Q(x)$ will also make the two functions $f(x)$ and $R(x)$ equal. Performing the division will give $Q(x)=x$ and

$$y = R(x) = \frac{p}{2}x^2 + \frac{3}{4}qx + s. \tag{12}$$

This is the equation of the parabola through the three bend points. It will also pass through the point $(0, s)$, a fourth point of the quartic. When $p=0$, there is no point of inflection and only one real bend point. In such a case $y=R(x)$ in (12) represents the equation of the secant of the quartic through the only real bend point and the point $(0, s)$.

To complete the part of this paper which concerns the influence of the coefficient p on the graphs of the quartics represented by (1), turn now to the case when $p \geqq 0$. The sketching of the quartic is then relatively simple. Its lower part is somewhat like a U askew according to the function of q. The curve is skew-symmetric to the y-axis along lines of slope q and lies above the tangent line $y=qx+s$. If a part of the curve has been located by points for positive values of x, then since the line $x=0$ is the diameter for chords of slope q such chords will locate corresponding points of the quartic to the left of the y-axis.

5. Certain tangents to the quartic. As a final topic, consider the tangents to the quartic from points on its pseudo-diameter, the line $x=0$. From the equation of the tangent given in (2) it is seen that b, its y-intercept, is

$$b = -3x_1^4 - px_1^2 + s. \tag{13}$$

This is independent of the coefficient q. Therefore, *a one parameter family of quartics as in* (1) *with fixed values for the coefficients p and s, but with q as an arbitrary parameter, has the property that all the tangents to the curves of this family at the points where $x=x_1$ will have the same y-intercept.*

A maximum value of b in (13) greater than $b=s$ will exist only when $p<0$. It is

$$b_{\max} = s + \frac{p^2}{12}.$$

The two tangents that have this y-intercept have their points of contact where $6x_1^2 = -p$. They are the two tangents of inflection.

From every point on the y-axis where $y \leqq s+p^2/12$ one, two, three or four tangents can be drawn to the quartics of (1) for which $p<0$. From the points on the y-axis between $(0, s)$ and $(0, s+p^2/12)$ four tangents can be drawn. From the point $(0, s)$ three tangents can be drawn. One of these is the tangent-secant line given in (3). The other two have their points of contact where $3x_1^2 = -p$. These two points lie on the secant line $y=qx+s-2p^2/9$. This secant will meet the quartic again at the two points where $3x_1^2 = -2p$. It turns out that the tangents at the two latter points are parallel to the two inflection tangents. From all points on the y-axis below the point $(0, s)$ two tangents can be drawn. However, at the point $(0, s-p^2/4)$ only one tangent exists. This is the double tangent given in (4).

The slopes of the two tangents of inflection are given by the expression

$$q \pm \left(\frac{-2p}{3}\right)^{3/2}.$$

If the slope of one inflection tangent is zero, then the slope of the other is $2q$.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California
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A GENERALIZED INTEGRAL TEST FOR CONVERGENCE OF SERIES

MORGAN WARD, California Institute of Technology

The following useful generalization of the familiar Maclaurin-Cauchy integral test for convergence of real series deserves to be better known. It is apparently due to G. H. Hardy,* who made a redundant hypothesis on $f(t)$. The integrals may be taken either in the sense of Riemann or in the sense of Lebesgue.

THEOREM. *Let $f(t)$ be a complex-valued function of the real variable in the interval $1 \leqq t < \infty$, such that $f'(t)$ exists and is integrable to $f(t)$ over any finite interval $1 \leqq t \leqq T$. Then if $\int_1^\infty f'(t)dt$ is absolutely convergent, the series $\sum_1^\infty f(n)$ and the integral $\int_1^\infty f(t)dt$ converge and diverge together.*

Proof: By Abel's partial summation formula, we have

$$\sum_{r=1}^{n} a_r b_r = A_n B_n - \sum_{r=1}^{n-1} A_r(b_{r+1} - b_r),$$

where $A_r = a_1 + a_2 + \cdots + a_r$, $(r = 1, 2, \cdots, n)$.

Let $s_n = \sum_{r=1}^{n} f(r)$. Then on taking $a_r = 1$ and $b_r = f(r)$ in the summation formula, we find that

$$s_n = nf(n) - \sum_{r=1}^{n-1} r(f(r+1) - f(r)).$$

Now if $[t]$ denotes as usual the greatest integer in t, then

* G. H. Hardy: Proc. London Math. Soc. (2), vol. 9, 1910, pp. 126–144.

$$r(f(r+1)-f(r)) = \int_r^{r+1} [t]f'(t)dt.$$

Also

$$nf(n) - 1\cdot f(1) = \int_1^n \frac{d}{dt}(tf(t))dt,$$

or

$$nf(n) = f(1) + \int_1^n f(t)dt + \int_1^n tf'(t)dt.$$

On substituting these expressions into the formula for s_n, simplifying and transposing, we obtain the formula

$$s_n - \int_1^n f(t)dt = f(1) + \int_1^n (t-[t])f'(t)dt.$$

Now $|(t-[t])f'(t)| < |f'(t)|$. Hence the infinite integral $\int_1^\infty (t-[t])f'(t)dt$ is convergent, and

$$\lim_{n\to\infty}\left(s_n - \int_1^n f(t)dt\right) \text{ exists.} \tag{1}$$

Now assume that the integral $\int_1^\infty f(t)dt$ is convergent. Then $\lim_{n\to\infty}\int_1^n f(t)dt$ exists. Hence by (1), $\lim_{n\to\infty} s_n$ exists; that is, the series $\sum_1^\infty f(n)$ is convergent.

The converse result is a little more troublesome. Assume that $\sum_1^\infty f(n)$ converges. Then

$$\lim_{n\to\infty} f(n) = 0, \tag{2}$$

and by (1),

$$\lim_{n\to\infty}\int_1^n f(t)dt \text{ exists.} \tag{3}$$

Now $f(T) = f(1) + \int_1^T f'(t)dt$. But since $\int_1^\infty f'(t)dt$ converges, $\lim_{T\to\infty}\int_1^T f'(t)dt$ exists. Hence $\lim_{T\to\infty} f(T)$ exists, so that by (2),

$$\lim_{t\to\infty} f(t) = 0. \tag{4}$$

Now

$$\left|\int_1^T f(t)dt - \int_1^{[T]} f(t)dt\right| = \left|\int_{[T]}^T f(t)dt\right| \leqq \max_{[T]\leq t\leq T} |f(t)|\,(T-[T])$$
$$< \max_{t\geq[T]} |f(t)|.$$

Hence by (4)

$$\lim_{T\to\infty}\left(\int_1^T f(t)dt-\int_1^{[T]} f(t)dt\right)=0.$$

But

$$\lim_{T\to\infty}\int_1^{[T]} f(t)dt$$

exists by (3). Hence $\lim_{T\to\infty}\int_1^T f(t)dt$ exists; that is $\int_1^\infty f(t)dt$ is convergent.

As an example, suppose that $f(t)=t^{-1}e^{-i\mu\log t}$, μ real. Then $f'(t)=0(1/t^2)$ and the conditions of the theorem are met. But

$$\int_1^T f(t)dt=\frac{i}{\mu}(e^{-i\mu\log T}-1).$$

Hence $\int_1^\infty f(t)dt$ diverges. Therefore $\sum_1^\infty 1/n^{1+i\mu}$ diverges.

Again, suppose that $f(t)=e^{it^\alpha\theta}/t^\beta$, where α and θ are real, and Rl $\beta>\alpha>0$, $\theta\neq 0$. Then $f'(t)$ is continuous and of order $t^{-1-\mu}$, where $\mu=$Rl $\beta-\alpha$, in the range $1\leqq t<\infty$. Hence the conditions of the theorem are met. Now the infinite integral $\int_1^\infty f(t)dt$ is easily seen to converge on making the change of variable $s=t^\alpha$. Hence the infinite series $\sum_1^\infty e^{in^\alpha\theta}/n^\beta$ converges. In particular then, if β is real, we see that the two real series

$$\sum_1^\infty\frac{\cos n^\alpha\theta}{n^\beta}\quad\text{and}\quad\sum_1^\infty\frac{\sin n^\alpha\theta}{n^\beta}$$

both converge if $\beta>\alpha>0$ and $\theta\neq 0$.

The ordinary integral test is included as a special case if we use Lebesgue integrals; for if $f(t)$ is real, continuous and tends to zero steadily, $f'(t)$ exists almost everywhere and $f(t)=\int_1^t f'(s)ds+f(1)$. Since $|f'(t)|=-f'(t)$, the hypotheses of the theorem are evidently satisfied.

GEOMETRY OF THE SQUARE ROOT OF THREE

C. S. Ogilvy, Trinity College, Hartford, Conn.

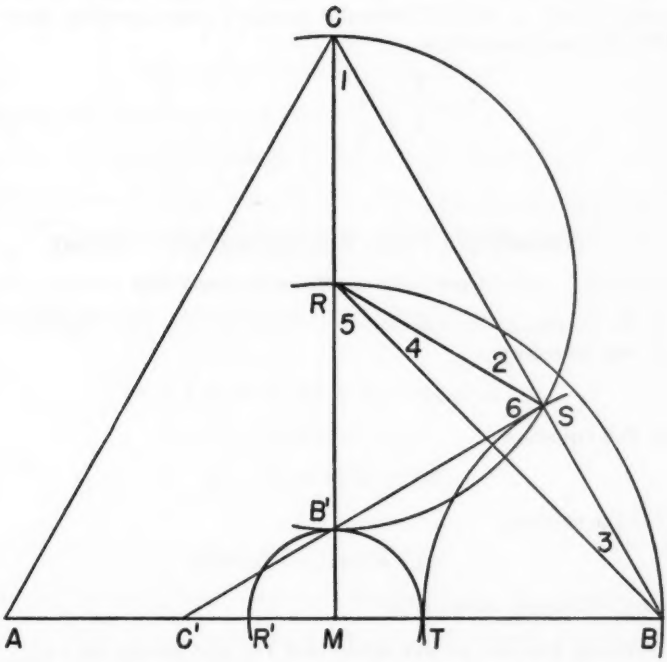
That the diagonal of a square is incommensurable with its side and the quotient is representable by the continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$$

is easy to prove geometrically. The corresponding fact that the altitude of an equilateral triangle and half its side are incommensurable and the quotient representable by

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cdots}}}}$$

is not quite so self-evident. A purely geometric proof follows.



Referring to the figure, lay off $MR=MB$. MB cannot be contained twice in MC, because $BC=2MB>MC$. Therefore the first quotient is 1 and the first remainder is $RC=r_1$.

Draw an arc with R as center and RC as radius. $\angle 1=\angle 2=30°$. Therefore $\angle RSB=150°$. But $\angle 3=15°$. Hence $\angle 4=15°$ and $RS=SB$. With B as center and BS as radius, draw an arc cutting BM at T. Then $BT=BS=SR=RB'=RC=r_1$. Now r_1 cannot be contained twice in BM; for if it were, it would be contained 4 times in CB; that is, $4r_1 \leqq CB$, an impossibility because $3r_1=(CR+RS+SB)>CB$. On the other hand, r_1 must be contained at least once in BM, since $BM=MR$ and $r_1<MR$. Therefore the second quotient is 1 and the second remainder is $MT=r_2$.

Since $MB'=MT$, an arc may be drawn with center at M, through T and B', cutting MA at R'. Draw SB', cutting AB at C'. Now $\angle 4+\angle 5=60°$ and

$RB'=RS$. Therefore triangle $RB'S$ is equiangular, and $\angle 6=60°$. Since $\angle 2=30°$, triangle BSC' is a 30°–60° right triangle. Hence $BC'=2BS=2BT$, and $C'T=TB=r_1$. Now triangle $MB'C'$ is similar to triangle MBC. $MR'=r_2$ is contained in MC' only once, as above, and hence only twice in TC', with remainder $R'C'$. Therefore the third quotient is 2 and the third remainder is $R'C'$.

The procedure with triangle $MB'C'$ will be exactly that used on triangle MBC. Thus the process is non-terminating, producing for subsequent quotients alternately 1 and 2, which is what is meant by the assertion that the ratio MC/MB can be expressed as

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cdots}}}}.$$

CONSECUTIVE CUBES WITH DIFFERENCE A SQUARE

VICTOR THÉBAULT, Tennie, Sarthe, France

1. E. P. Starke, in his solution to Problem E 702, this MONTHLY, vol. 53, 1946, p. 464, transforms

$$(n+1)^3-n^3=3n^2+3n+1=r^2 \tag{1}$$

into the Pell equation

$$(2r)^2-3(2n+1)^2=1, \tag{2}$$

and gives the solution

$$n_{k+4}=14n_{k+2}-n_k+6$$

containing all integer values of n.

2. Running, loc. cit., p. 465, notes that the first values for r are

$$1,\quad 13,\quad 181,\quad 2521,\quad 35113,\quad 489061,\quad 6811741,$$

and remarks that these numbers are all of the form $a^2+(a+1)^2$. He asks if the remark applies to all values of r satisfying (1).

The answer is affirmative, for, if we consider the equation

$$3n^2+3n+1=[a^2+(a+1)^2]^2,$$

we find easily that it reduces to

$$4X^2-3Y^2=1,$$

an equation of the same form as (2).

The numbers a satisfy

$$a_k=4a_{k-1}-a_{k-2}+1. \tag{3}$$

The property noted by Running is therefore general.

3. The values of a_k given by (3), the first being

$$a_1 = 0,\ a_2 = 2,\ a_3 = 9,\ a_4 = 35,\ a_5 = 132,\ a_6 = 494,\ a_7 = 1845, \cdots,$$

verify also

$$a_k + a_{k+1} + 2(a_k \cdot a_{k+1} + 1) = (a_k - a_{k-1})^2,$$

and

$$12a_k^2 + 12a_k + 9 = 3[(2a_k + 1)^2 + 2]$$

is a perfect square. For instance, if $n = 5$,

$$12(132)^2 + 12(132) + 9 = 210681 = 459^2.$$

4. Finally, (1) shows that r is the length of the side BC corresponding to the angle $A = 120°$ in a triangle ABC having for other sides the consecutive integers n and $n+1$.

In such a special triangle, *the number r measuring the largest side equals the sum of the squares of two consecutive integers a and $a+1$.*

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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FUNDAMENTAL IDENTITY OF VECTOR ALGEBRA

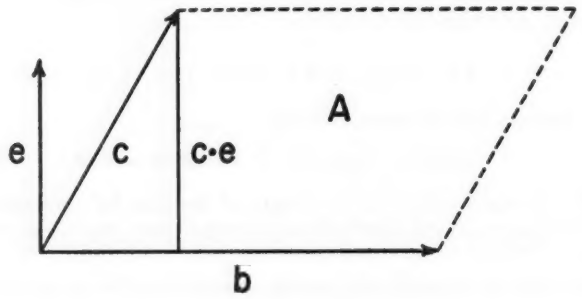
C. J. COE and G. Y. RAINICH, University of Michigan

One of the most valuable features of vector analysis as applied to geometry and physics is the fact that the discussion and results can be free from any reference to a particular coordinate system. This makes it desirable to define the vector operations in geometric form and only develop the corresponding coordinate formulas later as aids in the application. In this way the question of the invariance of these quantities under transformation of coordinates does not arise.

However, simple geometric proofs of the fundamental identity,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{1}$$

would seem to be lacking, since the proofs offered in the texts are extremely various and usually unsatisfactory from the above point of view. The following geometric proof is suggested, as resting directly on the geometric definitions of the operations involved and as clearly showing the geometric significance of the various terms.



Consider first the case in which **a** is a unit vector **e**, coplanar with **b** and **c** and perpendicular to **b** on the same side as **c**. Inspection of the figure, in which the plane of the paper is that of **b**, **c**, **e** shows that $\mathbf{e}\times(\mathbf{b}\times\mathbf{c})$ has the direction and sense of **b** and thus that the equality,

$$\mathbf{e} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\cdot\mathbf{e})\mathbf{b} \tag{2}$$

is true in direction and sense. Furthermore, the length of the left member is the area A of the parallelogram formed on **b** and **c**, while the length of the right member is the product of the altitude and base of this parallelogram. Thus equation (1) holds for $\mathbf{a}=\mathbf{e}$, since the second term of its second member vanishes in this case.

Likewise, in the case where **a** is a unit vector **f**, coplanar with **b** and **c**, but perpendicular to **c** on the same side as **b**, we have merely to replace **e** in equation (2) by **f** and interchange **b** and **c** to find,

$$\mathbf{f} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{f}\cdot\mathbf{b})\mathbf{c}, \tag{3}$$

and thus equation (1) is satisfied for $\mathbf{a}=\mathbf{f}$, since the first term of its second member vanishes in this case. And finally, if **a** is a unit vector **g**, perpendicular to both **b** and **c**, equation (1) holds since every term is zero.

Except in the trivial case in which **b** and **c** are parallel, any vector **a** is of the form $\mathbf{a}=k\mathbf{e}+l\mathbf{f}+m\mathbf{g}$, where k, l, m are scalars, and it follows that equation (1) holds in every case, since the equation involves **a** linearly and has been seen to hold for $\mathbf{a}=\mathbf{e}$, **f** and **g**.

THE PROBLEM OF A NON-VANISHING GIRDER ROUNDING A CORNER

NORMAN MILLER, Queen's University

In the usual problem of a girder being carried around a corner from one passageway to another, the girder is etherialized into a line segment (fig. 1). The problem gains in interest as well as in reality if we suppose the girder to have positive width. Moreover, the geometry to which it gives rise extends beyond the "practical" problem of the girder.

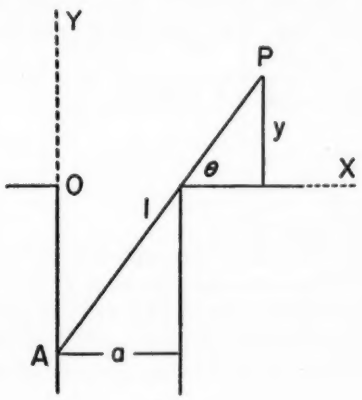


Fig. 1

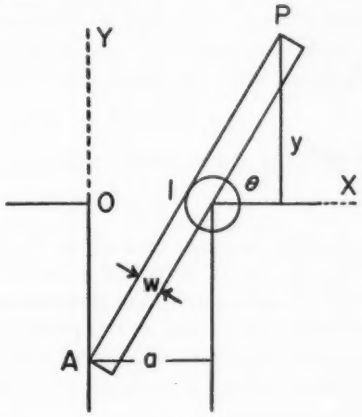


Fig. 2

Denote by w and l the width and length of the girder, supposed square, and by a the width of the passage from which the girder is being moved. We take $w<a<l$. Then, with the notation of fig. 2, the problem is to find the maximum of y where

$$y = l \sin\theta - (a - w \csc\theta) \tan\theta$$
$$= l \sin\theta - a \tan\theta + w \sec\theta.$$

Since

$$dy/d\theta = l \cos\theta - a \sec^2\theta + w \sec\theta \tan\theta,$$

the maximizing value of θ must satisfy

$$l \cos^3\theta - a + w \sin\theta = 0. \tag{1}$$

Attention to the graphs of the separate terms of this equation shows that it has exactly one root between 0 and $\pi/2$ and one between $-\pi/2$ and 0. If $w=0$, these two roots are equal in absolute value, but otherwise unequal.

The root of (1) which lies between 0 and $\pi/2$ gives the maximum value of y and hence the minimum width of the horizontal passageway which will allow the girder to turn the corner.

The root of (1) between $-\pi/2$ and 0 has also some interest. The further discussion will be concerned not with the problem of a physical girder but with the curve which is the complete locus of the point P. This locus will be defined as follows: A variable straight line touches a fixed circle of radius w and intersects in a variable point A a fixed straight line which is at a distance a from the centre of the circle. The locus in question is that of a point on the moving line which is at a distance l from A, where $w<a<l$. In case $w=0$ the locus is a Conchoid of Nicomedes which (in fig. 1) is symmetrical about OX. In case $w>0$ the locus is a sort of distorted Conchoid, having two points of contact with the fixed circle and containing an unsymmetrical loop. The parametric equations of this locus, with the axes indicated in fig. 2, are

$$x = l\cos\theta, \qquad y = l\sin\theta - a\tan\theta + w\sec\theta,$$

which give the single quartic equation

$$(xy - wl)^2 = (x - a)^2(l^2 - x^2).$$

This equation becomes that of the Conchoid when $w=0$.

When the point A (fig. 2) is on the negative y-axis, the angle θ is positive and when A moves up to the positive y-axis θ becomes negative. Thus all values of θ between $-\pi/2$ and $\pi/2$ apply to points on the curve and a critical point occurs on the loop for one positive value of θ and for one negative value (fig. 3). For $w>0$ the abscissas of these points are different as well as the absolute values of their ordinates.

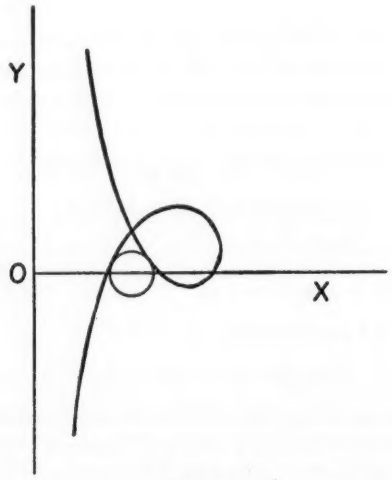


Fig. 3

Example. In the solution of the following numerical problem the maximizing root of (1) is easily identified: A $1\frac{1}{2}$ foot square beam 50 feet long is carried

through a passageway 12 feet wide into a lane at right angles to the passageway. Find the minimum width of the lane which will allow the beam to turn the corner while remaining horizontal.

Here $w=1.5$, $a=12$, $l=50$, and (1) reduces to

$$50\cos^3\theta - 12 + 1.5\sin\theta = 0. \tag{2}$$

This equation has the positive root given by $\cos\theta=3/5$, $\sin\theta=4/5$, for which we verify that $d^2y/d\theta^2<0$. This root gives the maximum value of y, which is found to be 26.5. The required minimum width of the lane is then 26.5 feet. The minimizing root of (2) is the negative value of θ given by $\cos\theta=0.64$, $\sin\theta=-0.79$, each correct to two figures. For this root it is easily verified that $d^2y/d\theta^2>0$. The corresponding point on the curve, the lowest point on the loop, has the approximate coordinates $(32, -23)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 847. *Corrected. Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b\sin A\left[b\cos A \pm (a^2 - b^2\sin^2 A)^{1/2}\right].$$

E 856. *Proposed by J. T. Hurt, Texas Agricultural and Mechanical College*

Let N be an integer of p digits. If the last digit is removed and placed before the remaining $p-1$ digits, a new number of p digits is formed which is $(1/n)$th of the original number. Find the most general such number N.

E 857. *Proposed by M. S. Knebelman, Washington State College*

Evaluate the s by s determinant whose element in the $(i+1)$st row and $(j+1)$st column is $d^{m+j}x^{n+i}/dx^{m+j}$.

E 858. *Proposed by Henry Scheffé, University of California at Los Angeles*

Give an example of an even function with continuous derivatives up to order n, which has neither a maximum nor a minimum at $x=0$.

E 859. *Proposed by C. W. Trigg, Los Angeles City College*

If the faces of a hexahedron are equilateral triangles congruent to the faces of a regular octahedron, then the radii of the inscribed spheres are in the ratio 2:3.

E 860. *Proposed by Leo Moser, University of Manitoba*

Show that if all the faces of a polyhedron have central symmetry then it can be dissected by a finite number of plane cuts and the pieces fitted together to form a solid cube.

SOLUTIONS

Distribution of Suits in a Bridge Hand

E 798 [1948, 502]. Correction to editorial note.

E. M. Berry and Bart Park have pointed out an error in the formula given in the editorial note. The initial factor should be 4, 12, or 24 according as three, two, or none of the n's are equal.

Incircle of Equilateral Triangle

E 820 [1948, 317]. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If ABC is an equilateral triangle, and P is any point on the circumference of the inscribed circle, prove synthetically that $(PA)^2+(PB)^2+(PC)^2$ is constant.

I. *Solution by R. P. Stephens, University of Georgia.* If r is the radius of the circle on which P lies and R is the radius of the circumcircle of ABC, then

$$(PA)^2 = r^2 + R^2 - 2r \text{ (projection of } OA \text{ on } OP).$$

We find similar expressions for $(PB)^2$ and $(PC)^2$. Hence the sum of the squares is equal to

$$3r^2 + 3R^2 - 2r \text{ (sum of projections of } OA, OB, OC \text{ on } OP).$$

But OA, OB, OC, considered as vectors, will form a closed figure whose projection on any line will be zero. Therefore

$$(PA)^2 + (PB)^2 + (PC)^2 = 3r^2 + 3R^2 = \text{constant}.$$

This theorem is a special case of a more general theorem which is easily proved in the same way. It is: *If $A_1 \cdots A_n$ is any regular n-gon with center O and radius R, and if P is any point on a concentric circle of radius r: then $\Sigma(PA_i)^2 = n(r^2+R^2)$.*

II. *Solution by Leo Moser, University of Manitoba.* (*Editorial Note.* Although the following solution is analytic, its simplicity and the possibility of numerous

generalizations, cause it to merit consideration.)

Take $A: (1, 0, 0)$, $B: (0, 1, 0)$, $C: (0, 0, 1)$, $P: (x, y, z)$. The inscribed circle is the intersection of a sphere $x^2+y^2+z^2=c_1$ and a plane $x+y+z=c_2$. Hence

$$\begin{aligned}(PA)^2 + (PB)^2 + (PC)^2 &= (x-1)^2 + y^2 + z^2 + x^2 + (y-1)^2 + z^2 + x^2 + y^2 + (z-1)^2 \\ &= 3c_1 - 2c_2 + 3 = \text{constant.}\end{aligned}$$

III. *Solution by William Gustin, Indiana University.* We shall establish the considerably more general theorem:

Let $A_1, \cdots, A_n$ be n points in a euclidean space and let $k_1, \cdots, k_n$ be n real numbers whose sum is unity. The locus of a point P for which

$$\sum k_i(PA_i)^2 = k, \text{ a constant,}$$

is a sphere whose center is the centroid G of the points A_i with weights k_i, and whose radius r is given by

$$r^2 = k - \sum k_i(GA_i)^2.$$

We shall employ vector methods, regarding points in the space as vectors. Let the function f be defined at every point P in the space as follows:

$$f(P) = \sum k_i(PA_i)^2 = \sum k_i(P\cdot P - 2P\cdot A_i + A_i\cdot A_i).$$

Since $\Sigma k_i=1$, the point $G=\Sigma k_iA_i$ is the centroid of the points A_i with weights k_i, and we have

$$f(P) = P\cdot P - 2P\cdot G + \sum k_i(A_i\cdot A_i),$$

whence

$$f(P) - f(G) = (PG)^2. \tag{1}$$

This proves the theorem.

Also solved by Murray Barbour, Paul Brock, W. E. Byrne, William Douglas, Ragnar Dybvik, J. M. Feld, Frank Harary, B. A. Hausmann, Sam Kravitz, B. R. Leeds, D. W. Matlack, Norman Miller, Leo Moser (a second solution), C. S. Ogilvy, Margaret Olmsted, A. P. Rhodes, C. C. Richtmeyer, Joseph Rosenbaum, A. Sisk, Kirk Stewart, Sieh Su and the proposer. C. W. Trigg (four solutions).

Editorial Note. Gustin's generalization (stated for the plane case only) can be found as Cor. 3, p. 99, in M'Clelland's *The Geometry of the Circle*, Macmillan, 1891. Many elegant theorems can be derived from it. Thus we have:

(*A*) *$\Sigma k_i(PA_i)^2$ is a minimum when P coincides with the centroid G of the points A_i with weights k_i.*

(*B*) *If $A_1, \cdots, A_n$ are vertices of a regular polygon (polyhedron) of circumradius R, and if P is any point on a concentric circle (sphere) of radius r, then*

$$\sum (PA_i)^2 = n(R^2 + r^2).$$

This is obtained from Gustin's relation (1) by taking $k_1 = \cdots = k_n = 1/n$. and (for the plane case) is the generalization stated above by Stephens. This generalization was also obtained, in different manners, by Rosenbaum and Trigg. Trigg also established the generalization for three-space.

(*C*) *If P is any point on the circumcircle (circumsphere) of a regular polygon (polyhedron) $A_1 \cdots A_n$ of circumradius R, then*

$$\sum (PA_i)^2 = 2nR.$$

This follows immediately from (*B*) by taking $r = R$. This result, for the polygon, appears as Cor. 2, p. 99, in M'Clelland.

(*D*) *If a_1, a_2, a_3 denote the sides of a triangle $A_1A_2A_3$ of incenter I, inradius r, and area K, and if P is any point on the incircle, then*

$$\sum a_i(PA_i)^2 = 2rK + \sum a_i(IA_i)^2.$$

In Gustin's relation (1) take $k_i = a_i/s$, where $s = a_1 + a_2 + a_3$. Then *G* coincides with *I* and we have

$$\sum a_i(PA_i)^2 = sr^2 + \sum a_i(IA_i)^2.$$

But $sr^2 = 2rK$.

This generalization of the given problem appears as ex. 7, p. 102, in M'Clelland.

(*E*) Rosenbaum pointed out that it can be proved that *if P is a point on a circle concentric with a regular polygon $A_1A_2 \cdots A_n$, then*

$$\sum (PA_i)^{2s} = \textit{constant}$$

for $s = 1, 2, \cdots, n-1$. For a proof of this see problem 3774 [1938, 190]. A special case is given as ex. 6, p. 101, in M'Clelland, where we find a proof of the fact that

$$\sum (PA_i)^4 = 6nR^4$$

if *P* is on the circumcircle of the regular polygon.

The Unscrupulous Athletes

E 821 [1948, 364]. *Proposed by B. H. Brown, Dartmouth College*

A sports promoter hired nine not too scrupulous athletes, and formed three cross-country teams *A*, *B*, and *C*, of three men each, which he took on tour for a series of dual and triple meets. If a team could win, it always would; but a losing team could be bribed to run more slowly. Except for this failing, the athletes always ran as automata, and always finished in the same order, with no ties. (In a cross-country meet the winner receives 1 point, the next man 2 points, *etc.* The team score is the sum of the points received by its members, and low score wins.)

In dual meets, team A beat team B, B beat C, and C beat A, much to the annoyance of the promoter who unjustly accused the men of dishonesty.

In an honest triple meet, team A won, whereupon team C went to team B with the following dishonest but logical offer: "No conceivable dishonesty on the part of our C team can enable your B team to beat or even tie A; but if you will slow down, our C team can beat A, and we will divide the profits with you."

Determine completely the composition of each team in terms of the relative excellence of its members.

Solution by Fritz Herzog, Michigan State College. We first introduce the following notations: Let X and Y be any two of the three teams. Then $[XY_i]$ is to denote the number of those members of team X that are faster than the ith member (in order of speed) of team Y, and $[X_iY]$ is to denote the number of those members of team Y that are slower than the ith member of team X, for $i=1, 2, 3$.

Obviously

$$\begin{aligned} &0 \leqq [XY_1] \leqq [XY_2] \leqq [XY_3] \leqq 3, \\ &3 \geqq [X_1Y] \geqq [X_2Y] \geqq [X_3Y] \geqq 0. \end{aligned} \tag{1}$$

Also, $[XY]$ is to denote the number of those among the nine pairs, consisting of one member of team X and one member of team Y, for which the former is faster than the latter. We have

$$\sum [XY_i] = \sum [X_iY] = [XY], \tag{2}$$

where the summation is always taken over i from 1 to 3, and

$$[XY] + [YX] = 9. \tag{3}$$

Let $S_2(X, Y)$ denote the score of team Y minus the score of team X in the dual meet between these two teams, and let $S_3(X, Y)$ denote the same difference in the triple meet.

Now it is easily seen that

$$S_2(X, Y) = 2[XY] - 9. \tag{4}$$

Furthermore, if Z is the third team, $S_3(X, Y)=S_2(X, Y)+[ZY]-[ZX]$, and so, by (4) and (3),

$$S_3(X, Y) = 2[XY] - [YZ] - [ZX] = -2[YX] + [ZY] + [XZ]. \tag{5}$$

According to the conditions of the problem, the five quantities $S_2(A, B)$, $S_2(B, C)$, $S_2(C, A)$, $S_3(A, B)$, and $S_3(A, C)$ are to be positive. Hence, by (4),

$$[AB] \geqq 5; \qquad [BC] \geqq 5; \qquad [CA] \geqq 5; \tag{6}$$

and, by (5), $2[AB]-[BC]-[CA]\geqq 1$ and $-2[CA]+[BC]+[AB]\geqq 1$, whence by addition

$$[AB] - [CA] \geq 1; \tag{7}$$

from (7) and (6),

$$[AB] \geq 6. \tag{8}$$

Let k_i denote $[AB_i]$, for $i=1, 2, 3$. Then the k_ith member of team A is faster than the ith member of team B, and we conclude that

$$[CA_{k_i}] + [B_iC] \leq 3, \qquad i = 1, 2, 3, \tag{9}$$

where $[CA_0]$ is to mean zero. Addition of the three inequalities (9) in conjunction with (2) and (6) yields

$$\sum_i [CA_{k_i}] \leq 4. \tag{10}$$

We shall now investigate the various possibilities for the $k_i=[AB_i]$. In the first place, we show that the three inequalities $k_i \geq i$ cannot hold simultaneously. For, by (2), (1), and (10), we would then have $[CA]=\Sigma[CA_i] \leq \Sigma_i[CA_{k_i}] \leq 4$, which contradicts (6). Now, by (2), (8), and (1), we must have $\Sigma k_i = [AB] \geq 6$ and $0 \leq k_1 \leq k_2 \leq k_3 \leq 3$. This leaves for k_1, k_2, k_3 only two possibilities, namely, 2, 2, 2 and 0, 3, 3.

If $k_1=k_2=k_3=2$, then (10) gives $3[CA_2] \leq 4$, hence $[CA_2] \leq 1$. Therefore, by (2), (6), and (1), we must have $[CA_3]=3$. Since $k_i=[AB_i]=2$ for $i=1, 2, 3$, the result of the dual meet between teams A and B must be $AABBBA$; and since $[CA_3]=3$ all three members of team C are faster than the slowest member of A. Consequently, team C could slow down in such a way that the triple meet would result in the order $AABBBCCCA$, which would mean a tie (12 points each) for A and B. This contradicts the conditions of the problem.

Thus the only possibility left for k_1, k_2, k_3 is 0, 3. 3. Again (10) gives $2[CA_3] \leq 4$, hence $[CA_3] \leq 2$. Since $[AB]=6$, we have, by (7), $[CA]=5$. These facts, together with (1) and (2), yield $[CA_1]$, $[CA_2]$, $[CA_3]=1, 2, 2$. If these values of the $[CA_i]$ are substituted in (9) one obtains $[B_1C] \leq 3$, $[B_2C] \leq 1$, $[B_3C] \leq 1$, so that, by (6), $[B_1C]$, $[B_2C]$, $[B_3C]=3, 1, 1$. The nine values of $k_i=[AB_i]$, $[CA_i]$, and $[B_iC]$, obtained above, are seen to be realized only if the (honest) triple meet ends in the order

$$B\ C\ A\ C\ A\ A\ B\ B\ C. \tag{11}$$

We still have to convince ourselves that (11) is actually a solution of the problem. In the dual meets A beats B, B beats C, and C beats A by scores of 9 to 12, 10 to 11, 10 to 11, respectively. In the (honest) triple meet the scores of A, B, C, are 14, 16, 15, respectively. Team B can arrange in numerous ways to let C beat A in the triple meet (for instance, by having the three members of B finish in the last three places so that the scores of C and A would be those of the dual meet between them). However, no matter how the first two members of team C are slowing down they will always decrease the score of A by at least

the same amount as that of B; thus C is unable to cause B to beat or even tie A.

Also solved by Murray Barbour, B. B. Dressler, Roger Lessard, Leo Moser, S. T. Parker, C. W. Trigg, and the proposer.

Editorial Note. In connection with this problem see the proposer's paper "The Scoring of Athletic Contests," *Scientific Monthly*, vol. LXII, pp. 233–237. Following the proposer, let the situation of team A beating team B, B beating C, and C beating A be called the *ring paradox*, and let the situation in which a losing team may control the relative ranking of the others be called the *control paradox*. In most of the solutions to the given problem the solvers first obtained the fifteen instances of the ring paradox that exist for this problem. Of these only three are such that A wins in an honest triple meet. By elimination, only one of these, in turn, is an instance of the control paradox where C may beat A.

Four Distinct Integers

E 823 [1948, 365]. *Proposed by Max LeLeiko, Rutgers University*

Find four distinct non-zero integers a, b, c, d such that

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= (1/5)[(a+b)^2 + (a+c)^2 + (a+d)^2 + (b+c)^2 + (b+d)^2 + (c+d)^2] \\ &= (1/7)[(a+b+c)^2 + (a+b+d)^2 + (a+c+d)^2 + (b+c+d)^2] \\ &= (1/3)(a+b+c+d)^2. \end{aligned}$$

Solution by Margaret Olmsted, Augustana College, Illinois. The three relations are equivalent to $\sum a^2 = \sum ab$. If we set $c=a-b$, $b=a+b$ in this we get $3a=4b$. Therefore a set of solutions is obtained by taking $a=4x$, $b=3x$, $c=x$, $d=7x$. Since the equations are symmetric in a, b, c, d, any permutation of a solution is also a solution.

Not all solutions are thus obtained, however, as no x will yield the solution 1, 1, 1, 3.

Also solved by Murray Barbour, L. J. Burton, Daniel Finkel, Roger Lessard, Leo Moser, W. V. Parker, and the proposer.

Editorial Note. Parker found a set of solutions given parametrically by $a=r^2$, $b=s^2$, $c=(r\pm s)^2$, $d=r^2\pm rs+s^2$. This set, like that obtained above, does not contain all solutions of the equations.

This problem is equivalent to problem 4269 [1947, 480], and a general solution may yet be furnished when the solutions to the latter problem are published.

Tetrahedra Circumscribing a Paraboloid of Revolution

E 830 [1948, 427]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The six planes bisecting the adjacent dihedral angles around the base of a

tetrahedron, taken four by four, form fifteen tetrahedra circumscribed about a common paraboloid of revolution.

Solution by the Proposer. It is easy to show that the orthogonal projections of the vertex A of the tetrahedron $ABCD$ on the six planes bisecting the interior and exterior dihedral angles along the edges BC, CD, DB all lie on the plane through the midpoints of the edges AB, AC, AD. Therefore these six planes are all tangent to the paraboloid of revolution having its focus at A and its vertex at the midpoint of the altitude through A. This proves the theorem.

The analogous theorem for the plane may be similarly established.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4336. *Proposed by Orrin Frink, Pennsylvania State College*

Find the arc of fixed length $l>2a$, lying below the x-axis and joining the points $(-a, 0)$ and $(a, 0)$, which includes between itself and the x-axis an area of lowest possible center of gravity. This will be the form actually assumed by a weightless flexible arc supported at its ends if it is holding water.

4337. *Proposed unsigned*

If the numbers R_{nk} are defined by

$$\frac{1-z^2}{\sin \pi z}\prod_{k=2}^{n}\sin(\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk}z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

4338. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

For any given triangle ABC inscribed in a circle (O), there are three points α, β, γ on (O) such that the segments determined by the sides of angle BAC on the tangents to (O) at α, β, γ have α, β, γ, respectively, for their midpoints. Show that the orthocenter of triangle $\alpha\beta\gamma$ is the midpoint of BC.

4339. *Proposed by Paul Erdös, Syracuse University*

Prove that if $n=2^u(2k+1)$, $k>1$, then 2^n-1 has a composite factor congruent to 1 modulo n.

4340. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $f(n)$ be the number of distinct equivalence relations connecting n elements. Show that

$$f(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^n \frac{(k-j)^n}{j!(k-j)!} = \frac{n^n}{n!}\left(\frac{1}{0!}\right) + \frac{(n-1)^n}{(n-1)!}\left(\frac{1}{0!} - \frac{1}{1!}\right) + \cdots$$
$$+ \frac{(n-r)^n}{(n-r)!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^r \frac{1}{r!}\right) + \cdots,$$

and find an asymptotic formula for $f(n)$ as $n \to \infty$.

Note. For equivalence relations and their connection with partitions see, Birkoff and MacLane, *Survey of Modern Algebra*, pp. 159 ff.

SOLUTIONS

A Sphere Related to the Tetrahedron

4206 [1946, 341]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to k times the sum of the squares of the three opposite edges. Show that the sum of the squares of the distances from the four vertices to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k+1)]^2R^2 - 2k(2k+1)\Sigma,$$

where R is the radius of the circumsphere and Σ means the sum of the squares of the six edges. Consider particular cases.

*Solution by the Proposer.** 1. If we designate by a, a', b, b', c, c' the lengths of the edges BC, DA, CA, DB, AB, DC of the tetrahedron $T \equiv ABCD$, then the squares of the radii of the specified spheres (A, m), (B, n), (C, p), (D, q) are defined by

$$m^2 = k(a^2 + b'^2 + c'^2), \qquad n^2 = k(a'^2 + b^2 + c'^2),$$
$$p^2 = k(a'^2 + b'^2 + c^2), \qquad q^2 = k(a^2 + b^2 + c^2).$$

* Translated by W. E. Byrne, Virginia Military Institute.

Let ω and r be the center and radius of the sphere (ω) orthogonal to the spheres (A, m), (B, n), (C, p), (D, q). We have

$$\overline{A\omega}^2 - r^2 = k(a^2 + b'^2 + c'^2), \qquad \overline{B\omega}^2 - r^2 = k(a'^2 + b^2 + c'^2),$$
$$\overline{C\omega}^2 - r^2 = k(a'^2 + b'^2 + c^2), \qquad \overline{D\omega}^2 - r^2 = k(a^2 + b^2 + c^2),$$

from which it follows that

$$\overline{A\omega}^2 - \overline{B\omega}^2 = k[(a^2 - a'^2) - (b^2 - b'^2)], \cdots .$$

But, if G is the centroid of T, we have

$$\overline{AG}^2 = [3(a'^2 + b^2 + c^2) - (a^2 + b'^2 + c'^2)]/16,$$
$$\overline{BG}^2 = [3(a^2 + b'^2 + c^2) - (a'^2 + b^2 + c'^2)]/16,$$

whence

$$\overline{AG}^2 - \overline{BG}^2 = -[(a^2 - a'^2) - (b^2 - b'^2)]/4, \cdots .$$

Hence the quantities $\overline{A\omega}^2 - \overline{B\omega}^2$, $\overline{B\omega}^2 - \overline{C\omega}^2$, $\overline{C\omega}^2 - \overline{D\omega}^2$, $\overline{D\omega}^2 - \overline{A\omega}^2$ are proportional to the quantities $\overline{AG}^2 - \overline{BG}^2$, $\overline{BG}^2 - \overline{CG}^2$, $\overline{CG}^2 - \overline{DG}^2$, $\overline{DG}^2 - \overline{AG}^2$. We obtain thus* the

THEOREM. *The locus of the center ω of the sphere (ω), as k varies, is the line OG which joins the circumcenter and the centroid of T.*

Furthermore,†

$$\overrightarrow{O\omega} = -4k\overrightarrow{OG}.$$

2. Application of Stewart's theorem to the triangle AOG and the point ω gives

$$G\omega \cdot \overline{AO}^2 + O\omega \cdot \overline{AG}^2 = OG \cdot \overline{A\omega}^2 + O\omega \cdot \omega G \cdot OG,$$

or, by the last relation above,

$$\overline{A\omega}^2 = (4k + 1)R^2 - 4k \cdot \overline{AG}^2 + 4k(4k + 1)\overline{OG}^2.$$

If we observe that $\overline{OG}^2 = R^2 - \Sigma/16$, and if we use the previously stated formula for $\overline{AG}^2$, this reduces to

$$\overline{A\omega}^2 = (4k + 1)^2R^2 - k(a'^2 + b^2 + c^2) - k^2\Sigma.$$

Likewise

$$\overline{B\omega}^2 = (4k + 1)^2R^2 - k(a^2 + b'^2 + c^2) - k^2\Sigma,$$
$$\overline{C\omega}^2 = (4k + 1)^2R^2 - k(a^2 + b^2 + c'^2) - k^2\Sigma,$$
$$\overline{D\omega}^2 = (4k + 1)^2R^2 - k(a'^2 + b'^2 + c'^2) - k^2\Sigma.$$

* V. Thébault, Mathesis, 1932, pp. 223–228.

† V Thébault, Mathesis, loc. cit.

Addition of the above equations gives the desired result,

$$\overline{A\omega}^2 + \overline{B\omega}^2 + \overline{C\omega}^2 + \overline{D\omega}^2 = [2(4k+1)]^2R^2 - 2k(2k+1)\Sigma.$$

3. Particular cases. (i) If $k = -\frac{1}{4}$, ω coincides with G, and we have the known result

$$\overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2 = \Sigma/4.$$

(ii) If $k = -\frac{1}{2}$, ω coincides with the Monge point M of T, whence‡ we have the

THEOREM. *The sum of the squares of the distances from the Monge point of T to the vertices of T is equal to the square of the diameter of the circumsphere of T.*

(iii) If $k = \frac{1}{2}$, ω coincides with the center M_1 of the Longchamps sphere of T. The expression for

$$r^2 = \overline{A\omega}^2 - k(a^2 + b'^2 + c'^2) = (4k+1)^2R^2 - k(k+1)\Sigma,$$

reduces to§

$$r^2 = 9R^2 - 3\Sigma/4.$$

The Deltoid

4245 [1947, 232]. *Proposed by J. H. Butchart, Arizona State College*

The envelopes of two families of lines, PQ, PQ', making angles of $\pm 30°$ respectively with the tangents to a deltoid at their points of contact P are two deltoids, larger than the given one in the ratio $3^{1/2}:1$. Show that $PQ = PQ'$, where Q, Q' are the points of contact of PQ, PQ' with the respective envelopes, and that the angles between the cusp tangents of the envelopes and the included cusp tangent of the given deltoid are $\pm 10°$.

Solution by R. Goormaghtigh, Bruges, Belgium. We prove a more general result from which the proposed problem follows as a special case. Let P be a variable point on any plane curve Γ, C the center of curvature of Γ at P, ρ the radius of curvature MC, ϕ the angle formed by the tangent of Γ at P and a fixed direction Δ. Consider a straight line PQ making a constant angle α with the tangent to Γ at P, Q being the contact point of that straight line with its envelope (Q).

For any straight line l, invariably attached to the tangent and the normal to Γ at P, the normal to the envelope of l at the contact point passes through C. Hence Q is the projection of C on PQ. This shows that, for any curve Γ, the points Q and Q' corresponding to any two angles α and $-\alpha$ are such that $PQ = PQ'$. Further, if (C) is the evolute of Γ and C_1 is the center of curvature of (C) at C, the angle formed by QC and the tangent PC to (C) being also α, then the point of contact of QC with its envelope, i.e. the center of curvature of (Q), is the projection of C_1 on QC. Hence, the radius of curvature of (Q) at Q is

‡ V. Thébault, Gazeta Matematica (Bucarest), 1933, p. 86.

§ V. Thébault, Mathesis, loc. cit.

$$R = \rho \cos \alpha + \rho_1 \sin \alpha,$$

ρ_1 being the radius of curvature CC_1 of (C).

Let us now consider the case when Γ is any cycloidal curve (i.e. cycloid, epicycloid or hypo-cycloid*), whence

$$\rho = a \sin n\phi,$$

a being a constant. Then we have

$$\rho_1 = d\rho/d\phi = an \cos n\phi$$

and

$$R = a(\sin n\phi \cos \alpha + n \cos n\phi \sin \alpha);$$

or, if ϕ' is the angle $\phi+\alpha$ formed by the tangent to (Q) and Δ, and if β is the angle such that $\tan \beta = n \tan \alpha$, we find

$$R = a \frac{\cos \alpha}{\cos \beta} \sin n(\phi' - \alpha + \beta/n).$$

Therefore, *if Γ is a cycloidal curve of index n, the straight line passing through a variable point P of Γ and making with the tangent at that point a constant angle α envelopes a cycloidal curve of the same index; if* β is the angle such that $\tan \beta = n \tan \alpha$, the ratio r of similitude of the second cycloidal curve to the first is $\cos \alpha/\cos \beta$, and the orientation of the second differs from that of the first by an angle $\alpha-\beta/n$.

When, as in the present problem, Γ is a deltoid, $n=3$; if $\alpha = \pm 30°$, then $\beta = \pm 60°$, $r = 3^{1/2}$, $\alpha-\beta/n = \pm 10°$. In the general case, when α is chosen so that $\tan \alpha = n^{-1/2}$, then $r = n^{1/2}$.

Arithmetic Progressions

4257 [1947, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any arithmetic progression show that the difference (sum) of the product of n consecutive terms and the product of n other consecutive terms is always divisible, provided n is even (odd), by the sum of the greatest and least of the terms. Prove also the corollary: If a and b are positive integers and if $a+b+1=p$ is prime, then

$$a!b! \pm 1 \equiv 0 \pmod{a + b + 1},$$

the sign being + or − according as a and b are even or odd.

Solution by P. A. Clement, Student, University of California at Los Angeles. For any two integers r and s we have $r \equiv -s \bmod (r+s)$, $r+d \equiv -(s-d)$, $\cdots$, $r+(n-1)d \equiv -[s-(n-1)d] \bmod (r+s)$. Then, by multiplication

* See Yates, A Handbook on Curves and Their Properties, 1947, pp. 81–85, 126.—Ed.

$$r(r+d)\cdots[r+(n-1)d] \equiv (-1)^n s(s-d)\cdots[s-(n-1)d] \bmod (r+s).$$

This is equivalent to the proposed problem. One notes that the two sets of consecutive terms need not be taken from the same progression (so long as the two progressions have the same common difference) and that the statement "by the sum of the greatest of either set and the least of the other" could replace the corresponding statement in the proposal.

Now in the above result put $d=1$, $s=a+b$, $r=1$, $n=x$. Then we have

$$a! \equiv (-1)^a(b+a)(b+a-1)\cdots(b+1), \bmod (b+a+1).$$

Multiplication by $b!$ gives

$$a!b! \equiv (-1)^a(b+a)! \equiv (-1)^{a+1} \bmod (b+a+1),$$

where the last step follows by Wilson's theorem since $a+b+1$ is prime. If $b+a>1$, then $b+a$ is even, and a and b are both even or both odd, and the congruence is thus equivalent to the proposed corollary. In the trivial case $a+b=1$, the sign is immaterial.

The corollary is equivalent to a problem in Uspensky and Heaslet, *Elementary Number Theory*, p. 157, no. 3.

Also solved by Murray Barbour, N. J. Fine, William Gustin, Roger Lessard. C. R. Phelps, and the Proposer.

Infinite Series and Product

4259 [1947, 418]. *Proposed by Richard Bellman, Princeton University*

If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1+x^{n_k}} = x\prod_{k=1}^{\infty}(1+x^{n_k}), \qquad |x|<1,$$

show that except perhaps for order $n_k=2^k$.

Solution by Shih-fang Li, Yenching University, Peiping, China. From the hypothesis, $n_k \geqq 0$ for all k. Since

$$f_k(x) \equiv x^{n_k}$$

are all analytic for $|x|<1$, we know that the product

$$F(x) \equiv \prod_{k=1}^{\infty}(1+x^{n_k})$$

converges everywhere in the unit circle and is itself analytic there.

By the given equality, since $F(x)\neq 0$ for $|x|<1$, we have*

* For theorems employed in this solution see K. Knopp, *Theory and Application of Infinite Series*, pp. 437, 438.

$$\frac{F'(x)}{F(x)} = \sum_{k=1}^{\infty} \frac{f_k'(x)}{1+f_k(x)} = F(x). \tag{1}$$

Solving and using the condition $F(0)=1$, we obtain

$$F(x) = (1-x)^{-1} = \sum_{k=0}^{\infty} x^k. \tag{2}$$

On the other hand, we may expand $F(x)$ in a power series. The corresponding finite product is

$$\begin{aligned} P_s(x) \equiv \prod_{k=1}^{s}(1+x^{n_k}) &= 1 + x^{n_1} + x^{n_2} + x^{n_1+n_2} + x^{n_3} \\ &+ x^{n_1+n_3} + x^{n_2+n_3} + x^{n_1+n_2+n_3} + \cdots + x^{n_1+n_2+n_3+\cdots+n_s}. \end{aligned} \tag{3}$$

Comparing (3) with (2), we have, except for order, $n_1=1$, $n_2=2$, $n_3=4$, $n_4=8$, $\cdots$, and by induction $n_k=2^{k-1}$ (instead of 2^k as given in the proposal).

Also solved by P. T. Bateman, M. S. Klamkin, and the Proposer.

Upper Density of a Sequence

4268 [1947, 479]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of a's up to n, then the upper density is defined as $\overline{\lim} f(n)/n$ as $n\to\infty$.) Then for suitable t the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r}, \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many t with this property.

Solution by the Proposer. Consider the k sequences

$$a_1 + a_j,\ a_1 + a_2 + a_j,\ \cdots,\ a_1 + a_2 + \cdots + a_k + a_j; \quad j = k+1,\ k+2,\ \cdots.$$

Each of them has upper density greater than $1/k$ and is a translation of the sequence $a_{k+1}, a_{k+2}, \cdots$. Thus the integers represented by them cannot all be different. Hence we must have

$$a_1 + a_2 + \cdots + a_{l_1} + a_{j_1} = a_1 + a_2 + \cdots + a_{l_2} + a_{j_2}.$$

Assume (without loss of generality) $l_2>l_1$. Then

$$a_{j_1} + a_{l_1+1} + a_{l_1+2} + \cdots + a_{l_2} + a_{j_2}$$

as required. It is easy to see that a_{j_1} can be found in infinitely many ways.

The integers $\equiv 1 \pmod{k}$ show that the theorem is false if the density is $1/k$.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. By H. A. Simmons. New York, The Macmillan Company, 1948. 10+619 pages. $4.00.

This book covers the topics usually given in college algebra. It seems to be an attempt to make a textbook that the college freshman can read himself. In the preface the author states, "In experience, we have found that the danger of over-explaining in *College Algebra* is mild in comparison with the opposite danger. Consequently, instead of attempting merely to give rigorous proofs with a minimum of words, we have tried to give rigorous proofs and at the same time add the extra word, phrase, illustration, or example that serves to 'put the idea across' immediately to the student."

This is a very laudable aim. However the reviewer feels the result is unnecessarily wordy. For example, the chapter on exponents and radicals contains 36 pages, while most standard texts cover the same material in less than half that many. Negative exponents are defined twice; on page 152 and again on page 161.

The text seems to emphasize rules for doing particular things rather than general principles, for example, the three sets of exercises for story problems leading to linear equations in one unknown.

A feature of the book is the large number of well graded exercises. There are so many that the answers (given for odd numbered problems only) cover 34 pages. The function idea is stressed throughout the book and in fractions it is always emphasized that the variables cannot assume values to make the denominator zero.

J. A. WARD

Mathematics of Finance. By J. B. Linker and M. A. Hill, Jr. New York, Henry Holt and Company, 1948. 8+171+83 pages. $2.90.

This book fits into the established pattern of text-books on the Mathematics of Finance. One of the authors' stated purposes is to make students "rely more upon their own efforts than to depend upon the instructor to explain each new step." Insofar as this formidable task can be accomplished through careful, explicit presentation, together with strategic use of line diagrams and illustrative examples, this text should be successful.

The explanation of the amortization schedule and its construction is perhaps better than average. The treatment of the method of equation of value, on the other hand, should probably be more explicitly set forth.

The subject of annuities is developed by stages. The reader is led through

the cases devolving upon how the size of the payment period compares with the size of the conversion period, arriving finally at the so-called general annuity. Further classification is made under the headings ordinary annuity, annuity due, and deferred annuity.

The topic of approximate investors' rate by the average net income method (usually presented in the chapter on bonds) has been omitted.

The problems are abundant and carefully stated, even if somewhat lacking in variety.

There is a 31 page introductory treatment of probability and life insurance.

In a 12 page appendix, the author gives an adequate summary of the elementary algebraic tools used in the text.

G. F. ROSE

Intermediate Algebra for Colleges. By W. L. Hart. Boston, D. C. Heath and Co., 1948. 7+316 pages. $2.50.

This is the most recent addition to the author's series of textbooks on algebra intended for first year college students. It is the most elementary one of the series and is intended to be suitable for students who have forgotten practically all of the techniques of algebra learned in high school. The topics treated are those normally covered in three semesters of high school algebra and the book is designed either to prepare students for the study of college algebra or to serve as preparation for courses in other fields which require a knowledge of elementary algebra as a prerequisite.

In common with practically all books of this kind the emphasis is on problem solving and the development of techniques, and scant attention is given to the number system of algebra or to a logical foundation for algebra. Irrational numbers, for example, are not even mentioned in the first half of the book. However, the author gives an excellent presentation of elementary algebraic techniques and the book will no doubt prove to be useful in many colleges which feel the need for courses of this type.

H. P. EVANS

Higher Algebra. By W. L. Ferrar. New York, Oxford University Press, 1948. 6+315 pages. $5.00.

This book is a sequel to the author's *Higher Algebra for Schools.* Like its predecessor, it is addressed to a superior group of British students and is not adapted to the traditional algebra courses given in American colleges.

The book consists of ten chapters dealing with the following subjects: finite series, infinite series and approximations, complex numbers, difference equations and generating functions, theory of equations, partial fractions, inequalities, and continued fractions.

The topics are few in number but are treated in great detail, and some are developed at greater length than in comparable textbooks. Among these may be

mentioned the exposition of linear functions of a complex variable, usually reserved for works on function-theory; the existence theorems for partial fractions; and the inequalities associated with the names of Maclaurin, Tchebychef, Weierstrass, Hölder, and Minkowski.

The book contains an abundance of well-chosen illustrative examples and classified problems, and some excellent advice on how to attack difficult problems. It is well adapted to individual study and is recommended to ambitious students desiring to cultivate algebraic techniques and to those preparing for competitive examinations in mathematics.

LOUIS WEISNER

Algebra for College Students. By J. R. Britton and L. C. Snively. New York, Rinehart and Co., 1947. 11+529 pages. $3.00.

Somewhat more than half of this volume has been published separately under the title *Intermediate Algebra* and has already been reviewed (*cf.* this MONTHLY, vol. 55, p. 376 (1948)). The present review will confine attention to the remainder of the text.

Typical topics include approximate numbers, logarithmic and exponential equations, investment problems, determinants, probability. Complex numbers are at first associated with vectors, afterwards treated in polar form for the student with trigonometric training. The chapter on inequalities is unusually full—eighteen pages long. The theory of equations includes, along with the elementary topics, a derivation of Descartes' rule of signs and considerable attention to numerical methods such as Horner's and the method of successive approximations, but excludes the algebraic solution of cubic and quartic equations. In regard to mathematical induction, a proof utilizes three steps, of which the second is showing the inheritance property. Review lists which occasionally finished chapters early in the book are noticeably absent in the latter half.

In the preface we read: "efficiency in manipulation alone is not the aim of the book. The major stress is on the important underlying ideas." Nevertheless most of the exercises for the student seem to be routine applications of methods developed in the text. Little opportunity is provided the student to learn to reason in algebra.

As noted in the *Intermediate Algebra* review, the writing is smooth and flowing. In conformity with this style, new material is frequently introduced from an intuitive approach, with perhaps a more rigorous viewpoint being adopted later. This mixture occasionally produces disturbing logical complications.

R. A. GOOD

NEW BOOKS RECEIVED

Introduction to the Algebraic Geometry of a Plane. By J. W. Archbold. London, Edward Arnold, 1948. 8+300 pages. $6.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1947–48

Kappa Mu Epsilon, Mount St. Scholastica College

The year's theme for *Kansas Gamma* Chapter of *Kappa Mu Epsilon* was *The true value of science in the liberal arts program.* The following discussions were heard:

True value of science in the liberal arts program, by Sister Helen Sullivan, O.S.B.

Student obligation to evaluate the role of science, by Frances Knightley

Is full academic life possible without a correct understanding of the hierarchy of the branches of knowledge?, by Victoria Fritton

Mathematics as a logical system of thought with a unique position in the hierarchy of knowledge, by Jean Moran

Non-Euclidean geometries,—foundations and development, by Sister Marietta Leuken, O.S.B.

The contribution of the particular sciences to learning, by Elaine Carson and Carrie Nelle Bremmer

Constructions with ruler and compass—three classical problems, by Gloria Jaskowiak.

Kansas *Gamma* was host to the four Kansas Chapters of Kappa Mu Epsilon at Mt. St. Scholastica College. About forty active members of Kappa Mu Epsilon attended this meeting to discuss local fraternity problems.

Officers for 1948–49 are: President, Gloria Jaskowiak; Vice-President, Gertrude Harrison; Secretary, Noreen Hurter; Treasurer, Mary Alice Weir; Publicity Chairman, Frances Donlon; Chapter Musician, Mary Jane Martin; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Kappa Mu Epsilon, Bowling Green State University

The following papers were presented to the *Ohio Alpha* Chapter of *Kappa Mu Epsilon*:

The history of aims of the national and local organizations of Kappa Mu Epsilon, by Prof. F. C. Ogg

A phase of the relativity theory, by Prof. D. W. Bowman of the Physics Department

Mathematics in economics and social science, by Prof. L. F. Manhart

Mathematics in industry, by Prof. D. M. Krabill

Mathematics in business, by Prof. H. R. Mathias

Mathematics in chemistry, Lewis Miller of the Chemistry Department

Teaching of mathematics, by Prof. Martha Gesling of the Department of Education

Beauty in mathematics, by Prof. F. C. Ogg.

The officers for next year are: President, Dallas Henry; Vice-President, Gordon Domick; Secretary, Donna Stroh; Treasurer, Arthur Miller; Advisor, F. C. Ogg; Corresponding Secretary, H. R. Mathias.

Pi Mu Epsilon, University of Oklahoma

The *Oklahoma Alpha* Chapter of *Pi Mu Epsilon* heard the following papers during 1947–48:

The development of mathematics, by Dr. C. E. Springer

Hyperbolic functions, by Dr. Arthur Bernhart

Greek astronomy, by Mr. Dewey McKnelly

The life of Gauss, by Mr. Truman Wester

Mathematical oddities and paradoxes, by Mr. L. D. Gregory

The essence of relativity, by Mr. R. B. Deal.

At the annual *Pi Mu Epsilon* banquet, the principal speaker was Dr. Lawrence H. Snyder, Dean of the Graduate College, who spoke on *Heredity and modern life.*

Kappa Mu Epsilon, Central Michigan College of Education

Several social gatherings and regular meetings were held by the *Michigan Beta* Chapter of *Kappa Mu Epsilon* during 1947–48. Among the papers presented were:

Star solids, by Miss Gertrude Pratt

Mathematical paradoxes, by James Laux.

The officers for 1948–49 are: President, William Kumbier; Vice-President, Jarold Brown; Secretary, Frances Woodbury; Treasurer, George Germain; Corresponding-Secretary, Mr. D. A. Sudborough.

Pi Mu Epsilon, Oregon State College

The *Oregon Beta* Chapter of *Pi Mu Epsilon* held five evening meetings during the academic year 1947–48. Programs were as follows:

Semantics of relativity, by F. H. Young

Five regular solids, by J. L. Ericksen

Sequential analysis, by J. F. Kahn

The solution of calculus problems by means of soap films, by E. E. Adams, and H. F. White

Finite summations, by R. D. Stalley.

There were forty-four members initiated during the year.

The officers for 1947–48 were: President, N. B. Smith; Vice-president, L. R. Stark; Secretary, Miss Marjorie Sims; Treasurer, Prof. G. A. Williams.

The officers elected for 1948–49 are: President, LaVerne Rickard; Vice-President, P. E. Harper; Secretary, F. Illig; Treasurer, Prof. G. A. Williams.

Pi Mu Epsilon, Ohio State University

The *Pi Mu Epsilon* Chapter at Ohio State University heard the following talks during 1947–48:

Perron's solution of the problem of Kakeya, by Dr. J. W. T. Youngs of Indiana University

Infinity, by Dr. R. L. Swain

Sequences of integers, by Dr. W. Scott

Electronic digital computers, by Dr. H. D. Huskey (Bureau of Standards, Washington, D. C.).

The following officers were elected for the year 1947–48: Faculty adviser, Prof. E. J. Mickle; Director, W. M. Myers; Vice-Director, J. E. Adney; Secretary-treasurer, R. A. Dean.

Graduate Mathematics Club, Ohio State University

The following talks were presented to the *Graduate Mathematics Club* during 1947–48:

On the extension of transformations, by Dr. E. J. Mickle

Length and area, by Dr. L. C. Young (University of Capetown, South Africa)

The structure of abstract algebras, by Prof. G. Birkhoff (Harvard University)

Introduction to distance in an abstract space, by Dr. M. Frechet (Paris, France)

On integral extensions of a commutative ring, by Dr. Harold Charland

Generalization of theorems, by Dr. H. Blumberg

Geometry, retrospect and prospect, by Dr. M. Hall

Analytic methods in non-linear mechanics, by Dr. C. E. Sealander

Topology of measure spaces, by Dr. O. W. Rechard (University of Wisconsin)

Representation problem for Frechet surfaces, by Dr. J. W. T. Youngs (Indiana University)

The present state of electronic digital computers, by Dr. H. D. Huskey (Bureau of Standards, Washington, D. C.).

Kappa Mu Epsilon, Montclair State Teachers College

The *New Jersey Beta* Chapter of *Kappa Mu Epsilon* at Montclair State Teachers College heard the following talks given at monthly meetings:

Einstein's theory of relativity

A discussion of the mathematics convention at Atlantic City

Artillery mathematics

The trends of education in England, by Dr. D. R. Davis

Non-Euclidean geometry

Mathematical tricks, by Dr. I. Brune.

Special events of the year included a joint Christmas party with Sigma Phi Mu, the mathematics club, and a banquet given in honor of Dr. Howard Fehr, Professor of Mathematics, who is leaving to join the Mathematics Department of Columbia University Teachers College.

Officers for the year of 1948–49 are: President, Robert Lundquist; Vice-President, Erwin Winguth; Secretary, May Christensen; Treasurer, Gloria Senapole; Corresponding Secretary, George Kays.

Mathematics Club, Montclair State Teachers College

Papers presented to the *Mathematics Club, Sigma Phi Mu*, during the year were:

Semantics, by Dr. I. Brune
Construction of polyhedra, by Wilma Freeze
Topology, by Clifford Swisher
The abacounter, by Dr. N. Lazar of Columbia
Mathematics in life insurance, by G. C. Campbell
Atomic energy, by Dr. Alyea of Princeton University.

The social program consisted of a fencing match, the annual Christmas Party with *Kappa Mu Epsilon*, and the annual Picnic.

Officers for the year of 1948–49 are: President, June Boswell; Vice-President, Joan Alexander; Secretary, Evelyn Pass; Treasurer, Herbert Gebner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE JOURNAL OF SOUTHEASTERN RESEARCH

A new publication, *The Journal of Southeastern Research*, will provide Southeastern engineers and scientists their own medium for recording research results. The Technical Section of the new journal will present articles of interest to research experts only; its News Section will present factual reports for business executives who wish to keep informed about scientific progress in the region. The first issue was published in January, 1949.

Dr. W. G. Pollard, Director, Oak Ridge Institute for Nuclear Studies, is one of the fourteen Southeastern scientists who has been named to the Advisory Board of the Journal.

PERSONAL ITEMS

Professor Tibor Rado of Ohio State University is the first appointee to the new rank of University Research Professor. To accept the new appointment, he resigned from his position as chairman of the Department of Mathematics.

Mr. Rudolph Feige has been appointed to an instructorship at the University of Cincinnati.

Professor W. W. Gandy of Northwestern State College, Louisiana, has been appointed to an instructorship at Agricultural and Mechanical College of Texas.

Mr. W. T. Guy, Jr. of the University of Texas has received an appointment as graduate assistant at California Institute of Technology.

Mr. Joseph Hilsenrath of the Naval Ordnance Laboratory has accepted a

position as scientific educational advisor with the National Bureau of Standards.

Professor Walter Lycha of Wartburg College has been appointed to an instructorship at Augustana College.

Professor W. E. Milne, head of Department of Mathematics, Oregon State College, is on leave of absence and is spending the year at the Institute of Numerical Analysis.

Miss Margaret Owchar, formerly teaching assistant at the University of Minnesota, has been appointed to an instructorship at Rockford College.

Associate Professor E. D. Rainville of the University of Michigan is on sabbatical leave and is located at the California Institute of Technology.

Professor P. V. Reichelderfer of Ohio State University has been named Acting Chairman of the Department of Mathematics.

Mr. Abraham Rosenfeld has been appointed Training Officer, Physics and Mathematics, at the Ordnance School, Aberdeen Proving Ground, Maryland.

Dr. Andrew Sobczyk of Watson Laboratories has been appointed to an assistant professorship at Boston University.

Assistant Professor D. B. Sumner of Louisiana State University and Agricultural and Mechanical College has been appointed to a lectureship (temporary) at the University of Toronto.

Assistant Professor S. L. Thompson of Alabama Polytechnic Institute has been promoted to an associate professorship.

Dr. Annita Tuller of Hunter College has been promoted to an assistant professorship.

Dr. Herschel Weil of Brown University has accepted a position as a mathematician with the General Electric Company, Schenectady, New York.

Professor Emeritus W. S. Hall of Lafayette College died December 17, 1948 at the age of eighty-seven years. He was a charter member of the Association.

Dr. Frank Irwin, associate professor emeritus of the University of California and a charter member of the Association, died on December 25, 1948 at the age of eighty years.

Dr. W. D. MacMillan, professor emeritus of astronomy and mathematics, University of Chicago, died November 14, 1948 at the age of seventy-seven years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The thirty-second annual meeting of the Mathematical Association of America was held at Ohio State University, Columbus, Ohio, on Friday, December 31, 1948, in conjunction with the annual meetings of the American Mathematical

Society, the Association for Symbolic Logic, and the National Council of Teachers of Mathematics. About five hundred and seventy-eight persons attended the meetings, including the following three hundred and sixty-three members of the Association:

V. W. ADKISSON, University of Arkansas
J. E. ADNEY, JR., Ohio State University
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
H. H. ALDEN, Ohio State University
C. B. ALLENDOERFER, Haverford College
E. W. ANDERSON, Iowa State College
R. D. ANDERSON, University of Pennsylvania
R. V. ANDREE, University of Wisconsin
MAX ASTRACHAN, Antioch College
FRANK AYRES, JR., Dickinson College
W. L. AYRES, Purdue University
GLADYS F. BADGER, Roosevelt High School, Chicago
A. H. BAILEY, Georgia Institute of Technology
N. H. BALL, U. S. Naval Academy
F. R. BAMFORTH, Ohio State University
T. A. BANCROFT, Alabama Polytechnic Institute
GRACE M. BAREIS, Ohio State University
I. A. BARNETT, University of Cincinnati
JUNA L. BEAL, Butler University
H. M. BEATTY, Ohio State University
J. H. BELL, Michigan State College
W. J. BELLMER, University of Dayton
A. A. BENNETT, Brown University
THEODORE BENNETT, Marietta College
W. D. BERG, Kenyon College
Brother ALFRED BERNARD, Manhattan College
BARBARA B. BETTS, D. C. Heath and Co.
WILLIAM BETZ, University of Rochester
R. H. BING, University of Wisconsin
A. H. BLACK, Southern Illinois University
C. J. BLACKALL, University of Toledo
HENRY BLUMBERG, Ohio State University
L. M. BLUMENTHAL, University of Missouri
R. P. BOAS, JR., Math. Reviews
J. G. BOWKER, Middlebury College
M. G. BOYCE, Vanderbilt University
G. F. BRADFIELD, De Paul University
HELEN J. BRADLEY, University of Tennessee
A. T. BRAUER, University of North Carolina
RICHARD BRAUER, University of Michigan
H. E. BRAY, Rice Institute
H. W. BRINKMANN, Swarthmore College
J. R. BRITTON, University of Michigan
FOSTER BROOKS, Research and Development Board, U. S. Army
K. E. BROWN, University of Tennessee
R. S. BURINGTON, Bureau of Ordnance, Navy Department
I. W. BURR, Purdue University
HERBERT BUSEMANN, University of Southern California
L. E. BUSH, College of St. Thomas
JEWELL H. BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota
S. S. CAIRNS, University of Illinois
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
V. B. CARIS, Ohio State University
A. B. CARSON, Army Air Forces Institute of Technology
J. W. CELL, North Carolina State College
R. V. CHURCHILL, University of Michigan
B. B. CLARK, University of Michigan
J. A. CLARKSON, Tufts College
MARY D. CLEMENT, University of Miami
ESTHER A. COMPTON, Cumberland College
J. A. COOLEY, University of Tennessee
W. F. CORNELL, Bowling Green State University
N. A. COURT, University of Oklahoma
R. R. COVEYOU, Oak Ridge National Laboratories
W. H. H. COWLES, Pratt Institute
JANE S. CRONIN, Princeton University
H. E. CRULL, Butler University
J. C. CURRIE, Alabama Polytechnic Institute
H. B. CURRY, Pennsylvania State College
WAYNE DANCER, University of Toledo
VIOLET B. DAVIS, University of Toledo
Rev. L. A. V. DECLEENE, St. Norbert's College
C. H. DENBOW, U. S. Naval Postgraduate School
A. H. DIAMOND, Oklahoma A. & M.
R. P. DILWORTH, California Institute of Technology
H. L. DORWART, Washington & Jefferson College
J. E. DOTTERER, Manchester College
Rev. W. C. DOYLE, Rockhurst College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, Tulane University

E. L. EAGLE, University of Arkansas
E. D. EAVES, University of Tennessee
P. D. EDWARDS, Ball State Teachers College
SAMUEL EILENBERG, Columbia University
H. P. EVANS, University of Wisconsin
G. M. EWING, University of Missouri
A. B. FARNELL, Princeton University
F. D. FAULKNER, University of Michigan
WILLIAM FELLER, Cornell University
H. E. FETTIS, Air Material Command
F. A. FICKEN, University of Tennessee
N. J. FINE, University of Pennsylvania
C. D. FIRESTONE, Rutgers University
M. P. FOBES, College of Wooster
CLARENCE FORD, Male High School, Louisville
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
J. S. FRAME, Michigan State College
ORRIN FRINK, JR., Pennsylvania State College
W. A. GAGER, University of Florida
B. E. GATEWOOD, Army Air Forces of Technology
H. M. GEHMAN, University of Buffalo
Rev. F. J. GERST, Loyola University
B. P. GILL, College of the City of New York
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
B. C. GLOVER, Otterbein College
E. L. GODFREY, Defiance College
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
MICHAEL GOLOMB, Purdue University
A. W. GOODMAN, Rutgers University
RUTH E. GOODMAN, Duquesne University
S. T. GORMSEN, University of Florida
M. J. GOTTLIEB, Newark College of Engineering
S. H. GOULD, Purdue University
F. G. GRAFF, Oberlin College
L. M. GRAVES, University of Chicago
J. B. GREELEY, Utica College of Syracuse University
J. W. GREEN, University of California
L. J. GREEN, Case Institute of Technology
LAURA Z. GREENE, Washburn University
V. G. GROVE, Michigan State College
W. S. GUSTIN, Indiana University
V. H. HAAG, Hershey Junior College
B. F. HADNOT, University of Georgia
FRANKLIN HAIMO, Washington University
MARSHALL HALL, JR., Ohio State University
P. R. HALMOS, University of Chicago
FRANK HARARY, University of Michigan
B. T. HARRIS, The Macmillan Co.
W. L. HART, University of Minnesota
E. E. HASKINS, Fenn College
J. O. HASSLER, University of Oklahoma
G. E. HAY, University of Michigan
C. T. HAZARD, Purdue University
C. H. HEINKE, Capital University
R. G. HELSEL, Ohio State University
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
J. J. L. HINRICHSEN, Iowa State College
CLARICE HOBENSACK, Western Hills High School, Cincinnati
H. K. HOLT, Union College
D. L. HOLL, Iowa State College
CARL HOLTOM, Army Air Forces Institute of Technology
E. MARIE HOVE, Hofstra College
Rev. J. A. HRATZ, St. Ambrose College
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
P. M. HUMMEL, University of Alabama
W. R. HUTCHERSON, Northwestern State College of Louisiana
L. C. HUTCHINSON, Brooklyn Polytechnic Institute
M. A. HYMAN, Naval Ordnance Laboratories
S. J. JASPER, Kent State University
R. L. JEFFERY, Queen's University
E. D. JENKINS, Kent State University
WALTER JENNINGS, U. S. Naval Postgraduate School
A. W. JONES, Rensselaer Polytechnic Institute
B. W. JONES, University of Colorado
MARGARET E. JONES, Ohio State University
P. S. JONES, University of Michigan
MARK KAC, Cornell University
H. S. KALTENBORN, Memphis State College
SIDNEY KAPLAN, Naval Ordnance Laboratories
IRVING KAPLANSKY, Institute for Advanced Study
H. T. KARNES, Louisiana State University
CHOSABURO KATO, Denison University
M. W. KELLER, Purdue University
J. L. KELLEY, University of California
K. D. KELLY, Fenn College
A. J. KEMPNER, University of Colorado
J. R. F. KENT, Triple Cities College
S. C. KLEENE, University of Wisconsin

J. R. KLINE, University of Pennsylvania
P. A. KNEDLER, State Teachers College, Kutztown, Pa.
L. C. KNIGHT, JR., Muskingum College
L. A. KNOWLER, University of Iowa
D. M. KRABILL, Bowling Green State University
MAX KRAMER, University of Illinois
H. W. KUHN, Ohio State University
O. E. LANCASTER, Bureau of Ordnance, Navy Department
A. E. LAMPEN, Hope College
R. E. LANGER, University of Wisconsin
GILLIE A. LAREW, Randolph-Macon Woman's College
E. H. LARGUIER, Spring Hill College
H. D. LARSEN, Albion College
C. G. LATIMER, Emory University
V. V. LATSHAW, Lehigh University
W. I. LAYTON, Alabama Polytechnic Institute
J. S. LEECH, University of Chicago
JOSEPH LEHNER, Hydrocarbon Research, Inc.
A. J. LEWIS, University of Denver
F. A. LEWIS, University of Alabama
MARY BETH LIEBERKNECHT, Iowa State College
B. J. LOCKHART, U. S. Naval Postgraduate School
CHARLES LOEWNER, Syracuse University
L. L. LOWENSTEIN, Kent State University
W. C. LOWRY, Kent State University
C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MACLANE, University of Chicago
INGO MADDAUS, JR., Union College
C. G. MAPLE, North Texas State Teachers College
MORRIS MARDEN, University of Wisconsin
R. H. MARQUIS, Ohio University
W. T. MARTIN, Massachusetts Institute of Technology
MARGARET E. MARTINSON, Washburn University
J. R. MAYOR, University of Wisconsin
N. H. MCCOY, Smith College
S. W. MCCUSKEY, Case Institute of Technology
W. C. MCDANIEL, Southern Illinois University
P. E. MEADOWS, Carroll College
A. E. MEDER, JR., Rutgers University
H. E. MENKE, Heidelberg College
G. M. MERRIMAN, University of Cincinnati
E. J. MICKLE, Ohio State University
H. J. MILES, University of Illinois
D. D. MILLER, University of Tennessee
F. H. MILLER, Cooper Union
L. H. MILLER, Ohio State University
W. E. MILNE, Oregon State College
H. J. MISER, Williams College
B. E. MITCHELL, Millsaps College
C. N. MOORE, University of Cincinnati
F. R. MORRIS, Fresno State College
R. D. MORRISON, Oklahoma A & M
D. C. MORROW, Wayne University
MARSTON MORSE, Institute for Advanced Study
W. B. MOYE, Georgia Teachers College
C. E. MULLAN, Duquesne Light Co., Pittsburgh
J. R. MUSSELMAN, Western Reserve University
W. J. NEMEREVER, University of Michigan
GRETA NEUBAUER, University of Wyoming
C. V. NEWSOM, University of State of New York
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of Illinois
C. O. OAKLEY, Haverford College
G. G. O'BRIEN, Washington Missionary College
RUFUS OLDENBURGER, De Paul University
L. F. OLLMANN, Hofstra College
EMMA J. OLSON, Kent State University
MORRIS OSTROFSKY, Duquesne University
E. R. OTT, RUTGERS UNIVERSITY
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
GORDON PALL, Illinois Institute of Technology
W. V. PARKER, University of Georgia
H. C. PARRISH, Ohio State University
PHILIP PEAK, Indiana University
SALLIE E. PENCE, University of Kentucky
P. M. PEPPER, University of Notre Dame
MARY PETTUS, Lander College
O. L. PHILLIPS, Mississippi Southern College
A. E PITCHER, Lehigh University
J. C. POLLEY, Wabash College
G. B. PRICE, University of Kansas
D. W. PUGSLEY, Berea College
TIBOR RADO, Ohio State University
LEILA R. RAINES, Cornell University
E. D. RAINVILLE, University of Michigan
J. F. RANDOLPH, University of Rochester
S. E. RASOR, Ohio State University
M. O. READE, University of Michigan
L. M. REAGAN, University of Wichita
O. W. RECHARD, Ohio State University
J. K. RECKZEH, State Teachers College, Jersey City
MINA S. REES, Office of Naval Research

P. V. REICHELDERFER, Ohio State University
ERIC REISSNER, Massachusetts Institute of Technology
C. N. REYNOLDS, West Virginia University
R. F. RINEHART, Research and Development Board, U. S. Army
FRED ROBERTSON, Iowa State College
L V. ROBINSON, University of South Carolina
V. N. ROBINSON, U. S. Naval Academy
W. J. ROBINSON, Centre College
L. D. RODABAUGH Southern Illinois University
P. C. ROSENBLOOM, Syracuse University
ARTHUR ROSENTHAL, Purdue University
M. F. ROSSKOPF, Syracuse University
S. A. ROWLAND, Ohio Wesleyan University
RAPHAEL SALEM, Massachusetts Institute of Technology
CHARLES SALTZER, Case Institute of Technology
R. G. SANGER, Kansas State College
A. C. SCHAEFFER, Purdue University
ROBERT SCHATTEN, University of Kansas
S. A. SCHELKUNOFF, Bell Telephone Laboratories
EDITH R. SCHNECKENBURGER, University of Buffalo
K. C. SCHRAUT, University of Dayton
E. W. SCHREIBER, Western Illinois State Teachers College
VERYL G. SCHULT, Wilson Teachers College
H. M. SCHWARTZ, Brookhaven National Laboratories
C. E. SEALANDER, Ohio State University
C. L. SEEBECK, University of Alabama
WLADMIR SEIDEL, National Bureau of Standards
M. E. SHANKS, Purdue University
H. C. SHAUB, Washington and Jefferson College
C. N. SHUSTER, State Teachers College, Trenton
L. L. SILVERMAN, Dartmouth College
Sister MARY PAULA, Marygrove College
F. C. SMITH, College of St. Thomas
R. E. SMITH, College of William and Mary
W. S. SNYDER, University of Tennessee
ANDREW SOBCZYK, Boston University
T. H. SOUTHARD, Wayne University
C. E. SPRINGER, University of Oklahoma
G. W. STARCHER, Ohio State University
E. P. STARKE, Rutgers University
H. E. STELSON, Michigan State College
R. C. STEPHENS, Knox College
GUY STEVENSON, University of Louisiana
B. M. STEWART, Michigan State College
RUTH W. STOKES, Syracuse University
M. H. STONE, University of Chicago
R. B. STONE, Purdue University
E. B. STOUFFER, University of Kansas
E. G. SWAFFORD, U. S. Naval Academy
OTTO SZASZ, University of Cincinnati
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
W. C. TAYLOR, University of Cincinnati
R. T. TEAR, Rensselaer Polytechnic Institute
H. P. THIELMAN, Iowa State College
L. O. THOMPSON, University of Detroit
R. M. THRALL, University of Michigan
G. L. TILLER, Utica College of Syracuse University
H. S. TONEY, Wilson College
LEONARD TORNHEIM, University of Michigan
MARIAN M. TORREY, Goucher College
A. W. TUCKER, Princeton University
J. L. ULLMAN, Stanford University
GILBERT ULMER, University of Kansas
E. P. VANCE, Oberlin College
HENRY VAN ENGEN, Iowa State Teachers College
H. E. VAUGHAN, University of Illinois
R. W. WAGNER, Oberlin College
G. L. WALKER, Purdue University
R. J. WALKER, Cornell University
S. E. WALKLEY, University of Illinois
J. L. WALSH, Harvard University
JEAN B. WALTON, University of Pennsylvania
W. R. WASOW, Swarthmore College
K. W. WEGNER, Carleton College
E. T. WELMERS, Bell Aircraft Corporation
INA W. WELMERS, University of Buffalo
F. J. WEYL, Office of Naval Research
E. A. WHITMAN, Carnegie Institute of Technology
P. M. WHITMAN, Johns Hopkins University
D. R. WHITNEY, Ohio State University
G. T. WHYBURN, University of Virginia
R. B. WILDERMUTH, Capital University
F. B. WILEY, Denison University
S. S. WILKS, Princeton University
W. L. WILLIAMS, University of South Carolina
C. O. WILLIAMSON, College of Wooster
R. L. WILSON, University of Tennessee
G. F. WOODSON, Wilberforce University
R. C. YATES, U. S. Military Academy
J. W. T. YOUNGS, Indiana University
J. H. ZANT, Oklahoma A & M
M. A. ZORN, Indiana University

The members of the mathematical organizations and their families were housed in Mack Hall and Canfield Hall. Meals were served in the Mack-Canfield dining hall.

The reception and tea held in the Faculty Club on Tuesday afternoon was well attended. On Wednesday evening, the School of Music of Ohio State University presented a program of music in University Hall Chapel. Miss Janice Murray was the vocalist, and Miss Eleanor Anawalt and Miss Olwen Jones of the University faculty formed a two-piano team.

To entertain the ladies present at the meeting, movies were shown on Wednesday afternoon. On Thursday afternoon, the Math Circle, composed of the wives and women staff members of the Mathematics Department, gave a tea for the visiting ladies.

A dinner for members of the mathematical organizations was held on Thursday evening in the Mack-Canfield dining room. Professor R. E. Langer was toastmaster. Dr. Harlan Hatcher, vice-president of Ohio State University, brought to the guests the greetings of the University on its 75th anniversary. Professor Saunders MacLane spoke on his experiences in Europe during the past year. Professor Marston Morse spoke also about his trip to Italy, and in particular about the Mathematical Congress held at Pisa. At the conclusion of the dinner, a resolution of thanks to the authorities of the University and the local members of the committee on arrangements was presented by Professor R. P. Dilworth and was enthusiastically adopted.

The sessions of the American Mathematical Society began on Tuesday, December 28 and continued through Thursday. On Tuesday evening, the twenty-second Josiah Willard Gibbs lecture was delivered by Professor Hermann Weyl of the Institute for Advanced Study, on the subject "Ramifications, old and new, of the eigenvalue problem." Professor Mark Kac of Cornell University spoke on Tuesday afternoon on "Probability methods in some problems of analysis and theory of numbers." On Wednesday afternoon, Professor A. S. Besicovitch of Cambridge University spoke on "Parametric surfaces" and Professor Lamberto Cesari of the University of Bologna spoke on "Area and representation surfaces." Professor Einar Hille of Yale University delivered his retiring presidential address on Wednesday morning on the topic "Lie theory of semi-groups of linear transformations."

Sessions of the Association for Symbolic Logic were held on Thursday and Friday. The National Council of Teachers of Mathematics held its Ninth Christmas Conference on Wednesday and Thursday. A feature of the conference was the simultaneous meeting of seventeen discussion groups on Thursday morning.

The Pi Mu Epsilon Fraternity held a breakfast meeting on Friday morning. The Kappa Mu Epsilon Fraternity held luncheon meetings on Tuesday and Wednesday.

The Mathematical Association of America held its sessions on Friday morn-

ing and afternoon in Room 306 Pomerene Hall. President L. R. Ford presided at the morning session and Vice-President C. B. Allendoerfer at the afternoon session. The Program Committee for this meeting consisted of J. S. Frame, Chairman, E. D. Rainville and Moses Richardson.

FIRST SESSION OF THE ASSOCIATION

"On the nature of applied mathematics," by Dr. R. S. Burington, Bureau of Ordnance, Navy Department.

"Mathematical aspects of the theory of viscous fluids," by Professor M. E. Shanks, Purdue University.

"Mathematical aspects of aero-elasticity," by Dr. E. T. Welmers, Bell Aircraft Corporation.

"Instructional aids in the teaching of Junior College mathematics," by Professor E. H. C. Hildebrandt, Northwestern University.

SECOND SESSION OF THE ASSOCIATION

"Modern operational calculus for undergraduates," by Professor J. R. Britton, University of Colorado and University of Michigan.

"A topic in the theory of differential equations," by Professor H. W. Brinkmann, Swarthmore College.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Thursday afternoon at 2 in Room 305 Pomerene Hall. Twenty-four members of the Board were present. Among the more important items of business transacted are the following.

The Board voted to approve the appointment of the following Nominating Committee for 1949: R. G. Sanger, chairman, W. E. Milne, W. M. Whyburn.

It was voted that one of the three representatives of the Association on the Policy Committee for Mathematics shall be the Secretary-Treasurer ex officio and that the other two representatives shall be elected by the Board hereafter for three year terms. A ballot resulted in the selection of these representatives: H. M. Gehman, ex officio (1949–1952), C. V. Newsom (1949–1951) and L. R. Ford (1949–1950).

The publication of an Index of volumes 1–55 of the MONTHLY was authorized. The Index is to contain a subject index of Mathematical papers and notes, a short author index, bibliographical material on club topics, miscellaneous official articles about the Association.

Upon recommendation of the Committee on Section Meetings, the Board voted that Sectional Governors shall be elected by a mail vote to be conducted by the Secretary-Treasurer of the Association, and that the Committee be authorized to determine the geographical boundaries of each Section and to allocate members to Sections for voting purposes. It was also voted that a maximum of $35 per year may be expended by the Secretary-Treasurer on be-

half of each Section toward the expenses of its meetings, including payment of travel expenses of an invited speaker.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

President L. R. Ford presided at the annual business meeting, which was held on Friday at 2:00 P.M. in Room 306 Pomerene Hall.

The tellers, L. E. Bush and P. D. Edwards, announced the election of R. E. Langer, University of Wisconsin, as President for the two-year term 1949–1950, and of C. B. Allendoerfer, Haverford College, and R. J. Walker, Cornell University, as Governors for the three-year term 1949–1951.

Announcement was made of the election by the Board of Governors of N. H. McCoy, Smith College, as Second Vice-President for the two-year term 1949–1950.

MEETING OF SECTION SECRETARIES

A meeting of Secretaries of the Sections of the Association was held on Thursday morning in Room 305 Pomerene Hall. Eighteen of the twenty-five Sections were represented. Professor W. V. Parker, Chairman of the Committee on Section Meetings, presided at the meeting.

The following topics were discussed: membership, election of sectional governors, programs of section meetings, finances, coordination of Association activities with those of elementary and secondary-school groups.

Professor Schneckenburger described some of the sources of new members by analyzing a group of recently elected members of the Association. A general discussion of programs led to many helpful suggestions from the representatives of the various sections. The discussion of methods of election of sectional governors and of finances led to the actions taken by the Board of Governors on these matters which are given above.

Professor E. H. C. Hildebrandt, President of the National Council of Teachers of Mathematics, urged the sections to provide a wider range of activities for teachers in the elementary and secondary schools. This was followed by a general discussion.

H. M. GEHMAN, *Secretary-Treasurer*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The seventh annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Washington Irving High School, New York, N. Y., on Saturday, April 24, 1948. Brother Bernard Alfred, Collegiate Vice-Chairman of the Section, presided at the morning session; Professor W. H. H. Cowles, Chairman of the Section, presided at the business meeting; and Mr. George G. Ross, High School Vice-Chairman, presided at the regular afternoon session.

One hundred and eighteen persons were present, including the following

sixty members of the Association: Brother Bernard Alfred, R. G. Archibald, W. D. Baten, W. C. Bornmann, Samuel Borofsky, C. B. Boyer, A. B. Brown, K. E. Brown, Jewell Hughes Bushey, Hobart Bushey, T. F. Cope, W. H. H. Cowles, J. G. Deutsch, J. N. Eastham, J. M. Feld, Daniel Finkel, R. M. Foster, K. G. Fuller, Harriet Griffin, George Grossman, Frank Hawthorne, G. C. Helme, R. E. Henry, E. Marie Hove, L. C. Hutchinson, R. A. Johnson, L. S. Kennison, E. R. Kiely, Edna Kramer-Lasser, B. R. Leeds, C. H. Lehmann, A. A. LePori, M. E. Levenson, Herman Levy, May H. Maria, A. L. Mayerson, Mary McKenna, F. H. Miller, A. J. Mortola, M. A. Nordgaard, P. B. Norman, Eugene Odin, J. K. Reckzeh, G. J. Ross, H. D. Ruderman, John Salerno, Charles Salkind, Aaron Shapiro, James Singer, E. R. Stabler, Mildred M. Sullivan, R. L. Vitale, H. E. Wahlert, Israel Wallach, Alan Wayne, Margaret C. Weeber, M. E. White, John Williamson, Sister Francis Xavier, R. C. Yates.

The Section elected the following officers for next year: Chairman, Professor R. A. Johnson, Brooklyn College; Collegiate Vice-Chairman, Professor T. F. Cope, Queens College; High School Vice-Chairman, Mr. H. D. Ruderman, Manhattan High School of Aviation Trades; Secretary, Professor James Singer, Brooklyn College; Treasurer, Mr. Aaron Shapiro, Midwood High School. Professor F. H. Miller, Cooper Union, was elected as the Sectional Governor.

Six papers were presented at the meeting.

1. *Practical methods of solving linear Diophantine equations*, by Professor A. B. Brown, Queens College.

If a linear Diophantine equation has a coefficient ± 1, the solutions are obvious. If not, a finite number of substitutions of the form $x = 1 \cdot x' + ay + \cdots + bz$ will either yield a coefficient ± 1, or disclose incompatibility.

For a system of equations, by taking linear combinations of left and right members of equations, an equation with a coefficient ± 1 may be obtained; otherwise the procedure for a single equation must be used. If an unknown has a coefficient ± 1 in an equation, it can be eliminated from the remaining equations, giving us a system with one fewer equations in one fewer unknowns to be solved. Again incompatibility shows up automatically. There are no formulas to be memorized.

2. *Basic values in junior high school mathematics*, by Miss Mary C. Rogers, Roosevelt Junior High School, introduced by the Secretary.

In a world where quantitative situations must be dealt with repeatedly and accurately, mathematical literacy is as important as the ability to read and write. The speaker held that the special functions of the junior high school are: (1) to provide an adequate and natural continuance of the work of the elementary school; (2) to correct all mathematical retardations and shortages existing among any of its pupils; (3) to provide an expanding and deepening experience with the problems of everyday living; (4) to strengthen and extend the foundations for subsequent experiences with mathematics. It was also stated that the mathematics curricula should include: (1) number and computation; (2) measurement and informal geometry; (3) constructions and interpretation of graphs; (4) an introduction to the functional core of algebra through formulas and equations; and (5) a generous application of the various phases of mathematics to the problem of everyday living. The speaker exhibited a number of charts, drawings, and solids made by students of her school.

3. *Generation, properties, and applications of some curves*, by Lt. Colonel R. C.

Yates, United States Military Academy.

Colonel Yates developed the conics as envelopes of creases formed by folding wax paper. He then discussed a simple ruler-compass construction for the center of curvature of all conics. This led to the consideration of evolutes and their application as caustics in the field of optics, with particular attention to the catacaustic of a circle for various point sources of light, and the diacaustic formed by light rays refracted through a plane surface. The idea of instantaneous centers of motion was then considered as a means of determining tangents to familiar curves such as the limaçon, strophoid, conchoid, and cycloid. The paper closed with remarks on the family of roses and their identification as special epitrochoids and hypotrochoids. Several models were used for illustration.

4. *Mathematical machines*, by Professor F. J. Murray, Columbia University, introduced by the Secretary.

Professor Murray stated that the characteristic aspects of current civilization appear in the utilization of scientific techniques which involve mathematics in a very fundamental way. But the range of application of mathematics is limited by our ability to solve complex problems and carry out computations. Thus the development of mathematical machinery is an important part of technical progress. Applied mathematics in general requires a considerable mathematical development beyond the pure theory, and mathematical machinery poses additional problems. It is similar to an increase in dimensionality. The study of mathematical machines is highly desirable for technical progress, its theoretical development has deep intellectual interest, and to the student of mathematics it offers a fascinating contact with the myriad of technical advances which distinguish our modern culture.

5. *The high school-college articulation group reports*, by Dr. Eugenie C. Hausle, James Monroe High School (introduced by the Secretary), and Professor J. H. Bushey, Hunter College.

Dr. Hausle discussed some of the recommendations submitted by the mathematics sub-committee on articulation between high school and college. Specific recommendations pertaining to methods of teaching and content of courses were dealt with. In particular, it was suggested that, beginning with the eleventh year, more and more of the responsibility for learning should be put upon the pupil. It was also recommended that the problems of coordination of mathematical teaching in the several divisions of the educational system could best be handled by a council for continuing the study of articulation between high school and college.

Professor Bushey discussed these matters from the point of view of the colleges. He pointed out the wide variation in the previous mathematical preparation of college freshmen and recommended a system of testing for placement purposes.

JAMES SINGER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The twenty-seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at the Illinois Institute of Technology, Chicago, Illinois, on Friday and Saturday, May 14–15, 1948. Professor John J. Corliss, Chairman of the Section, presided at all meetings.

There were 143 in attendance, including the following 69 members of the Association: M. L. Anthony, D. L. Arenson, H. G. Ayre, Ruth M. Ballard, H. R. Brahana, Winifred V. Berglund, S. F. Bibb, G. M. Bloom, F. R. Brown, E. L. Buell, Laura E. Christman, E. G. Comfort, J. J. Corliss, J. P. Esposito,

Sister Mary Esther, M. D. Eulenberg, I. K. Feinstein, L. R. Ford, Evelyn Frank, D. M. Friedlen, A. E. Gault, G. D. Gore, Cassie G. Greer, Madeleine Grenard, E. D. Hellinger, E. H. C. Hildebrandt, Sister Charlotte Holland, Rose Hornacek, E. C. Kiefer, W. C. Krathwohl, Luise Lange, R. E. Langer, Rose Lariviere, O. F. Latham, H. S. Levin, S. Levy, Leo Liolios, A. T. Lonseth, Sister Mariola, Karl Menger, K. W. Miller, G. E. Moore, C. W. Moran, E. J. Moulton, Mary W. Newson, E. P. Northrup, F. S. Nowlan, Grace M. Nolan, T. B. Ondrak, Gordon Pall, D. W. Pounder, Mary K. Rapp, W. T. Reid, Haim Reingold, Marian Rosenbeck, J. M. Sachs, E. W. Schrieber, H. W. Schwartz, Anice Seybold, Helen Sears, W. T. Scott, Beulah Shoesmith, H. A. Simmons, Albert Soglin, M. H. Stone, L. B. Stelling, M. E. Wescott, L. R. Wilsox, C. C. Wilson.

At business meeting held on Saturday morning the following were elected as officers for the coming year: Chairman S. F. Bibb, Illinois Institute of Technology; Vice-Chairman M. G. Moore, Bradley University; Secretary E. C. Kiefer, Millikin University. The meetings for next year are to be held on Friday and Saturday, May 13–14, 1949, at Bradley University in Peoria, Illinois.

On a motion by Professor L. R. Ford and seconded by Professor G. D. Gore, it was voted that the Illinois Section go on record as favoring, in principle, the encouragement of exceptional mathematical study and achievement in the high schools of Chicago, and that a committee be appointed by the Chairman of the Section to explore the possibilities of furthering these ends by contests and awards under the auspices of the Association. Chairman Corliss appointed Professor L. R. Ford as chairman of this committee.

A motion was passed instructing the Secretary of the Section to send a letter to Mrs. C. E. Comstock of Peoria expressing the feeling of the Section at the death of her husband, Professor C. E. Comstock, who was one of the founders the Section and an active member during his life.

The following papers were presented:

1. *Continued fractions related to the Riccati differential equation*, by Professor W. T. Scott, Northwestern University.

The author discussed a little known algorithm, first used by Euler and Lagrange, for obtaining from any Riccati differential equation,

$$N(x) - P(x) + Q(x)y^2 + y' = 0,$$

a continued fraction. A remainder formula was developed, thereby providing a test to determine whether or not such continued fractions actually converge to a solution of the differential equation. Finally, some special examples were considered.

2. *The use of differentials in thermodynamics*, by Professor Karl Menger, Illinois Institute of Technology.

The purpose of the paper was to bridge the gap between Calculus as taught in elementary mathematics courses and Calculus as applied by physicists. Especially in the field of thermodynamics the discrepancies baffle many beginners. The first law stipulates that the quantity of heat transferred to a gas is $pdv+du$ where p is the pressure, dv the change in volume, du the change of

the internal energy. Since this expression is not a complete differential, the original notation for the quantity of heat, dq, has been superseded by such symbols as $đq$, dq, and δq, which are not defined in Calculus. However, one can explain the nature of the "incomplete differential" in terms of the most elementary concepts of Calculus One merely has to notice that every thermodynamical process determines, and is characterized by five functions of the time, viz., $v(t)$, $p(t)$, $\theta(t)$, $u(t)$, $q(t)$, denoting the volume, the pressure, the temperature, and the internal energy of the gas, respectively, and the quantity of heat transferred to the gas between an initial moment t_0 and t. An equation of state implies the existence of a function Φ of three variables such that $\Phi[v(t), p(t), \theta(t)] = 0$ for every t. Then the first law can be written in the following differential form

$$\frac{dq}{dt} = p\,\frac{dv}{dt} + \frac{du}{dt} \quad \text{or} \quad q'(t) = p(t)\cdot v'(t) + u'(t)$$

for every t. An elementary integral form of the first law is

$$q(t) - q(t_0) = \int_{t_0}^{t} p(t)dv(t) + u(t) - u(t_0)$$

where the integral is a Stieltjes integral, that is, the limit of the sums

$$\sum_{i=0}^{n-1} p(t_{i+1}^{*})\left[v(t_{i+1}) - v(t_i)\right]$$

for subdivisions

$$t_0 < t_1 < \cdots < t_{n-1} < t_n = t, \qquad t_i \leqq t_{i+1} \leqq t_{i+1}^{*},$$

for which the maximum of the numbers $t_{i+1} - t_i$ approaches zero.

3. *Notes of the history of calculus with implications for our teaching*, by Professor E. D. Hellinger, Northwestern University.

It seems that investigations on calculus have always started with problems of integration rather than with problems of differentiation. This is shown in the classical Greek mathematics by Archimedes' evaluations of areas bounded by curves. His fundamental ideas were presented, especially those of his heuristic "method" and its relations to precise "geometric proofs." Again, at the beginning of modern mathematics one finds integration procedures; here, particularly, the basic ideas of Cavalieri were sketched. The subsequent development is based on the discovery of the close relation which can be established between the determination of the area beneath one curve and the slope of another one. Certain characteristic points in this line, leading to the proper differential and integral calculus, were indicated. It was pointed out that the pattern of historical development could be made useful for the modern teaching of calculus.

4. *Isaac Newton*, by Professor R. E. Langer, University of Wisconsin.

The speaker presented a brief biography of Newton, in which an attempt was made to project his scientific achievements against the background of the personal incidents of his life and the historical and sociological conditions of his time. In short sketches, the settings of his three great discoveries (the composite nature of white light, the calculus, the law of gravitation) relative to the achievements of his scientific predecessors were given. The episodes of the writing of the Principia, the ensuing collapse of his health, and finally of his later life in London as an official of the British mint were described.

5. *Objectives in the teaching of college mathematics*, by Professor F. S. Nowlan, University of Illinois.

Two main aims are considered in the teaching of college mathematics. These are: (1) Training

in precise thinking and a grasp of basic principles; (2) The acquisition of information and a mastery of certain technical skills. It is pointed out that a disproportionate stress has been placed upon the second of these to the almost total neglect of the first.

Various means are suggested whereby the first objective can be attained. In particular, the student should be required to express himself precisely and without ambiguity, in speech, in writing, and in symbolism. He should cease the misuse of the equality symbol, and the word "equals" must be used correctly. Also, the student must be trained to realize the postulational nature of mathematics, and in any study he must not only reason logically, but he must examine and question his initial assumptions.

6. *Mathematics in the Bachelor of Arts curriculum*, by Professor H. R. Brahana, University of Illinois.

For the last fifteen years it has been possible for freshmen to enter the College of Liberal Arts and Sciences of the University of Illinois without credits in high school mathematics. Recent changes in requirements for graduation make it necessary that a student have some mathematics beyond the elementary school level; the new requirement is satisfied by credit in algebra and geometry in high school, one year of each.

Close scrutiny of the small group of students affected by the changes in requirements reveals some unsuspected facts and also yields facts to support some opinions widely held. Briefly the more important facts are: (1) Almost all students entering the University have taken algebra and plane geometry in high school; (2) The rest have tried to do mathematics; (3) Those who have not done mathematics in high school disappear quickly from the University ; (4) Those who have not done mathematics in high school and who have low intelligence quotients disappear even more quickly; (5) Those who have not done high school mathematics are almost all deficient in knowledge of arithmetic.

Speculating about reasons why after fifteen years of not requiring mathematics for entrance we get so few who do not offer it, two reasons appear: (1) The children like to do mathematics; (2) Their teachers have put up a valiant resistance to those who would remove mathematics from the offerings of the high schools. Many high school teachers believe that the colleges have shirked a duty.

7. *Enrichment of mathematics offerings for juniors and seniors*, Professor M. H. Stone of The University of Chicago.

This discussion of the mathematics curriculum for juniors and seniors is directed to the need for incorporating recent mathematics advances in the course offerings, both as a phase of general education and as an aid to students intending to go on to graduate work in mathematics. The development of mathematics has resulted in a changed point of view as well as in strictly technical advances, both of which should be reflected in higher collegiate instruction. It is in algebra and geometry (including topology) that special efforts in designing new introductory courses at the junior and senior level seem to be needed. However, in the traditional courses some reflection of the vigorous growth of mathematics should also be expected.

E. C. KIEFER, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Thomas in St. Paul, Minnesota, on Saturday, May 8, 1948. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Professors R. L. Lokensgard, K. W. Wegner, L. S. Laws, and H. L. Turrittin (Chairman of the Section) presided at the respective sessions. Following the afternoon session, tea was served to the members and

guests by the wives of the mathematics staff of the College.

One hundred and nine persons attended the meeting, including the following forty-seven members of the Association: N. R. Amundson, H. M. Anderson, F. J. Arena, J. E. Bearman, Louise Beasley, Sister Ada Marie Boehm, K. H. Bracewell, R. W. Brink, M. D. Brown, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, Gladys Gibbens, Sister Seraphim Gibbons, H. W. Godderz, Charles Hatfield, Jr., J. S. Hill, T. W. Jackson, D. A. Johnson, G. K. Kalisch, W. S. Loud, C. C. MacDuffee, Kenneth May, W. H. McBride, W. R. McEwen, A. G. Montgomery, Sigurd Mundhjeld, M. J. Norris, F. R. Ohnsorg, J. M. H. Olmsted, G. C. Priester, P. A. Rognlie, L. W. Sheridan, F. C. Smith, Marion V. Smith, A. H. Speltz, A. G. Swanson, F. J. Taylor, Takashi Terami, Marian W. Thornton, Ella Thorp, H. L. Turrittin, K. W. Wegner, and Irene L. Wente.

The nominating committee, consisting of Professor W. H. Bussey, Chairman, Sister Seraphim Gibbons, and Professor Sigurd Mundhjeld, presented the following slate of nominees for the coming year: Chairman, W. P. McEwen, University of Minnesota, Duluth Branch; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, R. L. Lokensgard, Winona State Teachers College, H. M. Anderson, Gustavus Adolphus College, H. L. Turrittin, University of Minnesota. These were duly elected.

In accordance with instructions given at the 1947 annual meeting, the Secretary, acting for the Executive Committee, presented a proposed set of by-laws for the Section. On the motion by Professor Brink, seconded by Professor Bussey, the proposed by-laws were amended. They were then adopted, subject to approval by the Board of Governors, on the motion by Professor Brink, seconded by Professor Gibbens.

By invitation the Executive Committee, Professor C. C. MacDuffee delivered an address at the second morning session. The title of his address was *The Association Wants To Help*. He lent his support to the principle that the high school teacher should occupy an honored and well-paid position in his local community, for his work is fully as important and exacting as that of the physician or lawyer. In return, the teacher should be a genuine scholar, able to inspire in the minds of his students a high enthusiasm for the great achievements of the human mind and an ambition for productive scholarship. A present tendency to substitute in the name of "education for democracy" a cheaper type of education seems a cynical underestimation of the capacities of American youth. The sections of the Association should take a leading part in demanding a substantial curriculum in our public schools.

The following short papers were presented:

1. *The possibility of a simple iterative solution for simultaneous quadratics*, by Professor W. S. Loud, University of Minnesota.

A sufficient condition that the iterative process $x_{n+1}=f_1(x_n, y_n)$, $y_{n+1}=f_2(x_n, y_n)$ converge to a solution of the system of equations $x=f_1(x, y)$, $y=f_2(x, y)$ is that the matrix of the Jacobian of f_1 and f_2 with respect to x and y, evaluated at the solution, have its characteristic values both less than

unity in absolute value, and that the process be initiated sufficiently close to the solution. The concrete realization of this in the case of the system $x=y^3+A$, $y=x^2+B$ is that if the solution is (x_0, y_0), the relation $|4x_0y_0|<1$ holds. It follows as a consequence that not all systems of the latter type can be solved by the above process. The condition in terms of the parameters A and B are as follows: Let

$$g_1(A, B) = 256(A^3B^3 + A^3 + B^3) + 288AB - 27,$$
$$g_2(A, B) = 256(A^3B^3 + A^3B^3) + 160AB + 125.$$

Then, if g_1 and g_2 are both positive, and A and B both lie between $-5/4$ and $-3/4$, there are two real solutions which can be found by the above process. If g_1 and g_2 have opposite signs, there is one real solution which can be found by the process. In all other cases when g_1 and g_2 are both different from zero, no real solutions can be found by the process. If g_1 or g_2 is zero, there is a solution satisfying $|4x_0y_0|=1$.

By a suitable shift of origin and scale, the solution of almost any quartic equation can be reduced to the solution of the system $x=y^3+A$, $y=x^3+B$.

2. *Mathematics among the Maya Indians*, by Mr. Douglas Guy, Macalester College, introduced by Professor E. J. Camp.

The civilization of the Maya Indians of ancient Middle America produced the first known positional number system with a symbol for zero. The system was much like ours except that it used twenty as its base. In counting time, a further modification was made so that the second position above the units position indicated multiplication by 360 instead of 400. The Maya figured elapsed time from a hypothetical day of creation, about August 1, 3113 B.C. Thus, using commas to indicate that the figures separated by them are to be multiplied by 144000, 7200, 360, 20 and 1, the date *8, 14, 3, 1, 12* gave the exact number of days from the starting date and also approximated the number of years in the three highest "digits."

3. *On a certain type of wire puzzle*, by Professor Charles Hatfield, Jr., University of Minnesota.

The kind of wire puzzle dealt with was the "thread" type which consists of a loop of string (or if mechanical construction permits, loop of wire) around one or more parts of a wire configuration, the object of the puzzle being to disentangle the loop from the rest of it. It lends itself, both in construction and solution, to an analysis which utilizes the distinction between simply and multiply connected regions.

4. *An experimental course in mathematics*, Professor K. W. Wegner, Carleton College.

A two semester-hour course has been offered three times at Carleton in which the main objective was the development of clarity of thinking and expression. It was designed primarily for those who did not take any other college mathematics. In order to make it available to everyone, and since no attempt was made to cover a particular body of subject matter, no prerequisite was placed on the course.

Three main topics were included. The first of these was the topic of permutations, combinations, and probability. No formulas were developed, but each problem was solved from first principles. Stress was put on clarity of expression in describing solutions, whether oral or written. Number systems with bases different from ten were then studied. One period was devoted to playing and studying the game of Nim. The third main topic was from the theory of linear point sets. The class was given several definitions such as limit point, closed set, open set, derived set, etc. They were then given sets of various descriptions to test their understanding of the definitions. As a result of this experiment it is felt that such a course can go a long way in attaining the objective of the development of clarity in thinking and in expression.

5. *A study of the preparation of secondary school mathematics teachers in Minnesota*, by Mr. W. N. Nelmark, Mabel, Minnesota, introduced by the Secretary.

The twenty-one teacher training institutions in Minnesota varied in the number of credits required to graduate from a low of 120 semester hours to a high of 136 semester hours. The average was 126.2 semester hours. Practice teaching requirements averaged 4.3 semester hours, from a low of three semester hours to a high of eight semester hours.

One hundred fifty-six questionnaires were sent to college educators, mathematics professors, and high school mathematics instructors, of which over 77% responded. These people were asked to state which of certain topics in mathematics, education, and other subjects they considered of fundamental importance to a secondary school mathematics teacher, which they considered of considerable importance, which have some value, and which have no value.

Quite amazing were the results of the evaluation by high school mathematics teachers of the different mathematics subjects. Only the three subjects, college algebra, trigonometry, and analytic geometry, were considered of fundamental importance. Those of considerable importance were differential calculus, theory of equations, projective geometry and mathematics of investments. Those subjects having some value were integral calculus, spherical trigonometry, solid analytic geometry, history of mathematics, and mathematical statistics.

The general trend in the teacher preparation program in Minnesota seems to be for a sound general education, with broad fields of majors, increased certification requirements, five years of preparation with increased actual teaching experience, and raising the threshold standards of admission to teacher training institutions.

6. *Solution of a problem of Ramanujan*, by Mr. R. E. Graves, University of Minnesota, introduced by Professor R. W. Brink.

Ramanujan stated without proof that

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1} = \frac{1}{24} - \frac{1}{8\pi}.$$

This result was proved by Kac, in an unpublished paper, by use of the Poisson sum formula. At the time when Kac communicated his result to the author, he posed the problem of establishing Ramanujan's formula by methods which would have been available to the Indian mathematician. This paper establishes the formula by transforming

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1}$$

into

$$\sum_{n=1}^{\infty} ne^{-2\pi n},$$

thence into

$$\frac{1}{4}\sum_{n=1}^{\infty} \operatorname{csch}^2 n\pi,$$

and then summing this last series by elementary methods based upon the theorem of residues.

7. *A note on an inequality concerning subharmonic functions*, by Professor S. E. Warschawski, University of Minnesota, introduced by Professor Neal R. Amundson.

In the manner described below, the speaker developed an extension of a theorem of R. M. Gabriel. Let $u(x, y)$ be subharmonic in a region R; let C_1 and C_2 be simple closed rectifiable curves in

R, C_1 in the interior of C_2, such that it is possible to "roll" a circle of radius ρ_2 along C_2 on the "outside" of C_2, and a circle of radius ρ_1 along C_1 on the "inside" of C_1. Then $\int_{C_1} u ds \leq A \int_{C_2} u ds$, where A is a constant which depends only on ρ_2 and D/ρ_1 where D is the diameter of C_1. If C_2 is convex and C_1 is a circle, then $A = 2$.

8. *A note on cylindrical adsorption*, by Mr. C. E. Sanborn and Professor N. R. Amundson, University of Minnesota. (Mr. Sanborn was introduced by Professor Amundson.)

Consider a thin layer of thickness l and of infinite extent of adsorbent. A solution having a concentration c_0 of some solute is introduced on a circular area of radius R. The solute in solution on passing through the adsorbent is adsorbed. Letting the concentration be c, the amount adsorbed per unit weight q, the volume poured in v, the layer thickness l, and the fractional void volume α, then the following equation can be derived.

$$\frac{\partial c}{\partial r} + 2\pi r l \alpha \frac{\partial c}{\partial v} + 2\pi r l \frac{\partial q}{\partial v} = 0.$$

Assuming that the adsorption isotherm has the form $q = Af(c)$, A a constant, then

$$\frac{\partial c}{\partial r} + 2\pi r l [\alpha + Mf'(c)] \frac{\partial c}{\partial v} = 0.$$

The initial conditions are $c(0, r) = 0$ and $c(v, R) = c_0$. The solutions of this system were discussed, as well as the companion problem of the development of the chromatogram, i.e. solution with pure solvent.

9. *On the coefficients of certain multinomial expansions*, by Mr. F. J. Arena, North Dakota State College.

Tremblay has shown (*National Mathematics Magazine*, Vol. XI, 1937, p. 255) how to construct a generalized Pascal arithmetical triangle for the coefficients of multinomials of the form $(x+x^2 + \cdots + x^n)^m$. He writes unity n times for the first row. Then a term in any row is found by adding n terms of the above row. Begin with the term above the one sought and to it add the $(n-1)$ terms preceding it. With the use of such tables many problems in probability can be easily solved. For example, we can easily find the probability of getting 12 points with 4 dice.

Mr. Arena gives a rule for constructing such tables of the coefficients of the expansion of multinomials of the form $(1+x+x^2+ \cdots + x^n)^{-m}$. In the bottom row write 1 followed by several zeros. Then any term is found by adding the n terms preceding the term sought, subtracting from this the term directly below the term sought, and then changing the sign of this result. The following table, when read from the botton upward, gives the first few coefficients of the expansion of $(1+x+x^2+x^3)^{-m}$ for $m = 0, 1, 2, 3, 4, 5$.

1	−5	10	−10	10	−26	50	·	·	·
1	−4	6	−4	5	−16	24	·	·	·
1	−3	3	−1	3	−9	9	·	·	·
1	−2	1	0	2	−4	2	·	·	·
1	−1	0	0	1	−1	0	·	·	·
1	0	0	0	0	0	0	·	·	·

Thus

$$(1 + x + x^2 + x^3)^{-5} = 1 - 5x + 10x^2 - 10x^3 + 10x^4 - 26x^5 + \cdots .$$

10. *On the integral of a continuous function*, by Professor M. J. Norris, College of St. Thomas.

A method of proving that the Riemann integral of a continuous function on a closed interval exists without introducing the notion of uniform continuity was presented. The upper and lower integrals are shown to be additive. Then the difference of the upper and lower integrals, considered as a function of one end-point, is shown to have a derivative identically zero. It then follows that the upper and lower integrals are equal.

11. *Trends in the teaching of algebra*, by Professor K. H. Bracewell, Hamline University.

One hundred thirty-three replies received from a selected list of leading universities and colleges to whom a questionnaire was sent indicate a downward trend in pre-college preparation in algebra. Considerable variation exists among all higher institutions in both the amount of credit and content of courses in college algebra. Very few cover the chapters on theory of investment and infinite series. Permutations and combinations, probability and mathematical induction are in somewhat greater, but still limited, use. Sixty per cent of all schools reporting divide their algebra sections into elementary and advanced classes. No marked difference in practice was distinguished between universities and colleges.

12. *Concrete expression of mathematical ideas: an exhibit*, by Mr. E. J. Berger, Monroe High School, introduced by the Secretary.

The exhibit included about forty articles made by students at Monroe High School. Some of the articles illustrated properties of various algebraic curves, such as the focal properties of the ellipse and parabola. Others were measuring devices such as Jacob's staff, transit, clinometer, and sextant.

L. E. BUSH, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held at Purdue University, Lafayette, Indiana, on May 8, 1948.

Eighty-two persons attended the meeting including the following forty-seven members of the Association: J. L. Beal, Stanley Bolks, C. F. Brumfiel, Lee Byrne, G. E. Carscallen, K. W. Crain, H. E. Crull, Rev. H. F. De Baggis, M. W. De Jonge, R. H. Downing, Sister M. Virgilia Dragowski, F.O.S.F., P. D. Edwards, Ky Fan, E. L. Godfrey, Noel Gottesman, S. H. Gould, G. H. Graves, W. S. Gustin, Smith Higgins, Jr., Carl Holtom, H. K. Hughes, H. F. S. Jonah, P. S. Jones, M. W. Keller, E. L. Klinger, Florence Long, Sister Mary Ferrer McFarland, R.S.M., Karl Menger, G. T. Miller, P. M. Nastucoff, Paul Overman, Philip Peak, J. C. Polley, P. M. Pepper, C. K. Robbins, Arthur Rosenthal, A. E. Ross, G. X. Saltarelli, L. S. Shively, R. B. Stone, Raimond Struble, Anna K. Suter, G. L. Walker, M. S. Webster, A. M. Welchons, K. P. Williams and M. A. Zorn.

At the business meeting it was decided that the fall meetings which for the past several years have been held jointly with the Mathematics Section of the Indiana Academy of Science will be discontinued. The spring meeting of 1949 will be held at the University of Notre Dame.

P. D. Edwards, Ball State Teachers College, was elected Section Governor. Other officers elected at the meeting are: Chairman, H. E. Wolfe, Indiana Uni-

versity; Vice-chairman, A. E. Ross, University of Notre Dame. P. M. Pepper, University of Notre Dame, continues as Secretary. P. D. Edwards was appointed chairman of a committee of three, the other members to be chosen by him, to investigate the possibility of the Indiana Section compiling a report on the curricula of the colleges of Indiana similar to the report of a committee of the Michigan Section.

Professor Karl Menger of the Illinois Institute of Technology gave an interesting hour lecture entitled *Are Variables Necessary in Calculus?* Professor Menger's paper is to be published in the MONTHLY.

Professor P. S. Jones, University of Michigan, on invitation of the Section, gave a report entitled *Report of a Study of the High School Mathematics Prerequisite to Various College Curricula in Michigan College.* He described the activities and findings of the Committee on High School Mathematics of the Michigan Section of the Mathematical Association of America composed of Professors H. W. Alexander, C. C. Richtmeyer, and the speaker. Further information on this subject is presented in the February, 1949, issue of the MONTHLY in the paper by C. C. Richtmeyer.

The following papers were presented:

1. *On the use of a single axis, and of the unit circle in the teaching of trigonometry*, by Professor J. C. Polley, Wabash College.

The first part of the paper was a discussion emphasizing the lack of both utility and theoretical importance in the so-called vertical axis in the development of trigonometric theory, concluding with the opinion that it might better be abandoned in favor of a system in which the coordinates of points are defined relative to a single axis and a point thereon. The rest of the paper was devoted to a discussion on the more extensive use of the unit circle in trigonometry. In illustration the forms for the sine and cosine of the sum and difference of two angles, and those for the sum and difference of the sines and of the cosines of two angles were derived.

2. *On the teaching of determinants*, by Professor A. E. Ross, University of Notre Dame.

It was the purpose of the speaker to derive the usual properties of determinants from a set of assumptions connected as directly as possible with the solution of systems of linear equations. He considered an $n\times n$ matrix $A=(a_{ij})$ and the related systems of equations (I) $a_ix_j=\beta$. He showed that if a function $V(A)=V(\alpha_1\cdots\alpha_n)$ has the properties

$$V(\alpha_1\cdots c\cdot\alpha_k\cdots\alpha_n)=cV(\alpha_1\cdots\alpha_n) \tag{1}$$

$$V\left(\alpha_1,\cdots,\alpha_k+\sum_{j\neq k}c_j\alpha_j,\cdots,\alpha_n\right)=V(\alpha_1\cdots\alpha_n) \tag{2}$$

$$V(I)=V(\epsilon_1\cdots\epsilon_n)=1 \tag{3}$$

then

$$\begin{aligned}V(\alpha_1\cdots\alpha_n)x_k&=V(\alpha_1\cdots a_kx_k\cdots\alpha_n)=V(\alpha_1,\cdots,\textstyle\sum a_ix_i\cdots\alpha_n)\\&=V(\alpha_1\cdots\beta\cdots\alpha_n)=V_k.\end{aligned} \tag{II}$$

Thus if $V(A)\neq 0$, then system (I) has solutions x_i and $x_i=V_k/V$ (Cramer's rule). Following Artin, he proved the "product" formula, and, specializing one of the factors, showed that $V(\alpha_1\cdots\alpha_n)$ is the desired multilinear form with the correct rule of signs for the individual terms. The existence

of V with the properties (1), (2), and (3) is proved by induction. In teaching, one may employ areas of parallelograms and volumes of parallelopipeds (which do have properties (1), (2), (3)), together with (II) to derive an equivalent of Cramer's rule without the use of determinants in the usual elementary sense, and thus pave the way for the general geometrical theory.

3. *Determination of the area of a triangle from its sides*, by Professor E. L. Godfrey, Defiance College, Defiance, Ohio.

The presentation of the theory of simultaneous equations and determinants may well include a method of determining the area of a triangle from the equations of its sides, as well as that commonly given using its vertices.

4. *Exponent laws for integral powers*, by Professor M. A. Zorn, Indiana University.

The exponent laws for integral powers of the same base are derived by means of a modified induction principle.

5. *An introduction to a new theory of elementary complex geometry*, by Mr. E. L. Klinger, Purdue University.

To each point in the kth, three-dimensional complex space are assigned coordinates of the form $(x+ik, u+iv)$, or briefly (z, w), where x, u and v are real variables and k is any real constant. After a discussion of distances, it was shown that the equation of any line, except one lying in planes that are perpendicular to the z-axis, had the form $Ax+Bw+C=0$, where A, B and C are real or complex constants, except when $B=0$. In this case any line through (Z, W) may be represented by the system $z=x+ik$, $w=L(w)+ia$, if L and a are real constants and L an arbitrary one. Certain derived formulas involving the angle γ of intersection of two lines were discussed.

6. *Simplicially interlocking spheres*, by Professor William Gustin, Indiana University.

In an n-dimensional euclidean space let there be given $n+1$ closed spheres S_k such that the simplex T spanning the $n+1$ centers of these spheres is non-degenerate, and such that the simplex spanning any subset of the centers is covered by the spheres with those centers. According to a known theorem, due jointly to Knaster, Kuratowski, and Masurkiewicz, there exists a point common to all the spheres S_k and the simplex T. In this note such a point is found by elementary means.

7. *Short formulations of Boolean algebra, using ring operations*, by Dr. Lee Byrne, Purdue University.

Much interest has attached to recent formulations of Boolean algebras intended to emphasize their character as rings, and thus featuring especially ring operations. Most of these are relatively long, and Dr. Byrne's note was concerned with the question whether a simple formulation of this type might show appreciably more brevity. Leaving closure (and non-emptiness) assumptions tacit, he presented four "transformation" postulates, followed by ten theorems, which suffice to show the system to be a ring, a Boolean ring (i.e. one in which every element is idempotent), and a Boolean algebra (i.e., a Boolean ring with unit). The number of transformation axioms appears to be about two less than in previous versions with a similar approach.

8. *On the eigenvalues of symmetric kernels*, by Professor Ky Fan and Mr. Norman Haaser, University of Notre Dame.

Let the kernel $K(s, t)$ be real symmetric in $a \leqq s, t \leqq b$, and such that the classical Hilbert-Schmidt's theory is applicable. (I) Let ξ be a real number and let the eigenvalues λ_i of K be so arranged that $|\lambda_i-\xi| \leqq |\lambda_{i+1}-\xi|$, $(i=1, 2, \cdots)$. Then for any fixed integer $m>0$, $|\lambda_j-\xi|$ is the

greatest value which can be taken by the *G. L. B.* of the expression $(\|(\xi K-I)^m f\|/\|K^m f\|)^{1/m}$ when f is orthogonal to $j-1$ arbitrarily fixed functions. (II) From the case $m=1$, $j=1$ of (I) one obtains directly the inclusion theorem of D. H. Weinstein (*Proc. Nat. Acad. Sci.*, vol. 20, 1934, pp. 529–532. (III) If the eigenvalues of K are bounded from below and so arranged that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots$, then as limiting case $\xi = -\infty$ of (I), for any fixed even integer m, λ_j is the greatest value which can be taken by the *G. L. B.* of $(K^{m-1}f, f)/(K^m f, f)$ when f is orthogonal to $j-1$ arbitrarily fixed functions. If, in addition, K is positive definite, then it can be shown, as was proved by L. Collatz (*Matk. Zeitschr.*, vol. 46, 1940, pp. 692–708) and R. Iglisch (*Math. Ann.*, vol. 118, 1942, pp. 263–275), that the above characterization of λ_j holds for any integer m, even or odd, and $(K^{m-1}f, f)/(K^m f, f)$ is non-increasing with respect to m. (IV) If K, K', K'' are three real symmetric kernels such that

$$K(s, t) = \int_a^b K'(s, r)K''(r, t)dr,$$

and if their respective eigenvalues λ_i, λ_i', λ_i'' are so arranged that

$$|\lambda_i| \leqq |\lambda_{i+1}|, \qquad |\lambda_i'| \leqq |\lambda_{i+1}'|, \qquad |\lambda_i''| \leqq |\lambda_{i+1}''|,$$

then $|\lambda_{i+j+1}| \geqq |\lambda_{i+1}'| \cdot |\lambda_{j+1}''|$ holds for all $i, j \geqq 0$. This inequality implies that, for the composite kernel K, the series $\sum |\lambda_n|^{-1}$ converges. This is a particular case of a thorem due to Lalesco-Gheorghini (cf. Hille-Tamarkin, *Acta Math.*, vol. 57, 1931, p. 31).

P. M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 7, 1949.

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, May 7, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS, Kansas State College, Manhattan, April 2, 1949

KENTUCKY, Centre College, Danville, May 14, 1949

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, April 8–9, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, Spring, 1949

METROPOLITAN NEW YORK, Brooklyn College, April 9, 1949

MICHIGAN, Wayne University, Detroit, April 2, 1949

MINNESOTA, Gustavus Adolphus College, St. Peter, May 7, 1949

MISSOURI, University of Missouri, Columbia, April 9, 1949

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April 22–23, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Denton, April 8–9, 1949

UPPER NEW YORK STATE, University of Buffalo, April 30, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

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